‘Trigonometry is a sine of the times.’

–**Author Unknown**

MATHEMATICS
LESSON PLANS
GRADE 12 TERM 1

NATIONAL EDUCATION COLLABORATION TRUST (NECT)

Dear Teachers

This learning programme and training is provided by the National Education Collaboration Trust (NECT) on behalf of the Department of Basic Education (DBE). We hope that this programme provides you with additional skills, methodologies and content knowledge that you can use to teach your learners more effectively.

WHAT IS NECT?

In 2012 our government launched the National Development Plan (NDP) as a way to eliminate poverty and reduce inequality by the year 2030. Improving education is an important goal in the NDP which states that 90% of learners will pass Maths, Science and languages with at least 50% by 2030. This is a very ambitious goal for the DBE to achieve on its own, so the NECT was established in 2015 to assist in improving education.

The NECT has successfully brought together groups of people interested in education so that we can work collaboratively to improve education. These groups include the teacher unions, businesses, religious groups, trusts, foundations and NGOs.

WHAT ARE THE LEARNING PROGRAMMES?

One of the programmes that the NECT implements on behalf of the DBE is the ‘District Development Programme’. This programme works directly with district officials, principals, teachers, parents and learners; you are all part of this programme!

The programme began in 2015 with a small group of schools called the Fresh Start Schools (FSS). Curriculum learning programmes were developed for Maths, Science and Language teachers in FSS who received training and support on their implementation. The FSS teachers remain part of the programme, and we encourage them to mentor and share their experience with other teachers.

The FSS helped the DBE trial the NECT learning programmes so that they could be improved and used by many more teachers. NECT has already begun this scale-up process in its Universalisation Programme and in its Provincialisation Programme.

Everyone using the learning programmes comes from one of these groups; but you are now brought together in the spirit of collaboration that defines the manner in which the NECT works. Teachers with more experience using the learning programmes will deepen their knowledge and understanding, while some teachers will be experiencing the learning programmes for the first time.

Let’s work together constructively in the spirit of collaboration so that we can help South Africa eliminate poverty and improve education!

www.nect.org.za

CONTENTS

PROGRAMME ORIENTATION

Welcome!

The NECT FET Mathematics Learning Programme is designed to support teachers by providing:

- Lesson Plans
- Trackers
- Resource Packs
- Assessments and Memoranda
- Posters.

This Mathematics Learning Programme provides most of the planning required to teach FET Mathematics. However, it is important to remember that although the planning has been done for you, preparation is key to successful teaching. Set aside adequate time to properly prepare to teach each topic.

Also remember that the most important part of preparation is ensuring that you develop your own deep conceptual understanding of the topic. Do this by:

- working through the lesson plans for the topic
- watching the recommended video clips at the end of the topic
- completing all the worked examples in the lesson plans
- completing all activities and exercises in the textbook.

If, after this, a concept is still not clear to you, read through the section in the textbook or related teacher's guide, or ask a colleague for assistance. You may also wish to search for additional teaching videos and materials online. Some useful web links are listed at the end of each lesson plan.

Orientate yourself to this Learning Programme by looking at each component, and by taking note of the points that follow.

TERM 1 TEACHING PROGRAMME

1. In line with CAPS, the following teaching programme has been planned for FET Mathematics for Term 1:

Grade 10		Grade 11		Grade 12	
Topic	**No. of weeks**	**Topic**	**No. of weeks**	**Topic**	**No. of weeks**
Algebraic expressions	3	Exponents and Surds	3	Sequences and Series	3
Exponents	2	Equations and Inequalities	3	Functions (including inverses and logarithms)	4
Number Patterns	1	Number patterns	2	Finance, Growth and Decay	2
Equations and Inequalities	2	Analytical Geometry	3	Trigonometry	2
Trigonometry	3				
Total	11	Total	11	Total	11

* Note: CAPS amendments to be implemented in January 2019 require that in Grade 12, Euclidean Geometry be done in Term 1. The Grade 12 lesson plans reflect this. In order to ensure that you have the full set of topics for Grade 12, we have included the Topic of Finance at the back of the Grade 12 lesson plans. Finance is NOT done in Term 1.

2. Term 1 lesson plans and assessments are provided for eleven weeks for all three grades.

3. Each week includes 4,5 hours of teaching time, as per CAPS.

4. You may need to adjust the lesson breakdown to fit in with your school’s timetable.

LESSON PLAN STRUCTURE

The Lesson Plan for each term is divided into topics. Each topic is presented in exactly the same way:

TOPIC OVERVIEW

1. Each topic begins with a brief **Topic Overview**. The topic overview locates the topic within the term, and gives a clear idea of the time that should be spent on the topic. It also indicates the percentage value of this topic in the final examination, and gives an overview of the important skills and content that will be covered.

2. The **Lesson Breakdown Table** is essentially the teaching plan for the topic. This table lists the title of each lesson in the topic, as well as a suggested time allocation.
   For example:

	Lesson title	Suggested time (hours)
1	Revision	2,5
2	Venn diagrams	2,5
3	Inclusive and mutually exclusive events; Complementary and Exhaustive events	1,5
4	Revision and Consolidation	1,5

3. The **Sequential Table** shows the prior knowledge required for this topic, the current knowledge and skills to be covered, and how this topic will be built on in future years.
   - Use this table to think about the topic conceptually:
     - Looking back, what conceptual understanding should learners have already mastered?
     - Looking forward, what further conceptual understanding must you develop in learners, in order for them to move on successfully?
   - If learners are not equipped with the knowledge and skills required for you to continue teaching, try to ensure that they have some understanding of the key concepts before moving on.
   - In some topics, a revision lesson has been provided.

4. The **NCS Diagnostic Reports**. This section is potentially very useful. It lists common problems and misconceptions that are evident in learners' NSC examination scripts. The Lesson Plans aim to address these problem areas, but it is also a good idea for you to keep these in mind as you teach a topic.

5. The **Assessment of the Topic** section outlines the formal assessment requirements as prescribed by CAPS for Term 1 (page 54).

Grade	**Assessment requirements for Term 1 (as prescribed in CAPS)**
10	Project/ Investigation; Test
11	Project/ Investigation; Test
12	Project/ investigation; Assignment; Test

The assessments are included in the Lesson plans and Resource Pack for each grade.

Mathematics School-based Assessment Exemplars – CAPS. Grade 12 Teacher Guide.

Some of the Grade 12 assessments come from: *Mathematics School-based Assessment Exemplars – CAPS. Grade 12 Teacher Guide. DBE, Pretoria*

A team of experts comprising teachers and subject advisors from different provinces was appointed by the DBE to develop and compile the assessment tasks in this document. The team was required to extract excellent pieces of learner tasks from their respective schools and districts. The panel of experts spent a period of four days at the DBE developing tasks based on guidelines and policies. Moderation and quality assurance of the tasks were undertaken by national and provincial examiners and moderators to ensure that they are in line with CAPS requirements.

Mathematics School-based Assessment Exemplars – CAPS. Grade 12 Teacher Guide. DBE, Pretoria, p4

You can access this document from various sites, including:
https://www.education.gov.za/SchoolBasedAssessmentTasks2014/tabid/611/Default.aspx

6. The glossary of **Mathematical Vocabulary** provides an explanation of each word or phrase relevant to the topic. In some cases, an explanatory sketch is also provided. It is a good idea to display these words and their definitions or sketches somewhere in the classroom for the duration of the topic. It is also a good idea to encourage learners to copy down this table in their free time, or alternately, to photocopy the Mathematical Vocabulary for learners at the start of the topic. You should explicitly teach the words and their meanings as and when you encounter these words in the topic.

INDIVIDUAL LESSONS

1.. Following the **Topic Overview**, you will find the **Individual Lessons**. Each lesson is structured in exactly the same way. The routine within the individual lessons helps to improve time on task, and therefore, curriculum coverage.

2. In addition to the lesson title and time allocation, each lesson plan includes the following:

   A. ***Policy and Outcomes***. This provides the CAPS reference, and an overview of the objectives that will be covered in the lesson.

   B. ***Classroom Management***. This provides guidance and support as you plan and prepare for the lesson.

   - Make sure that you are ready to begin your lesson, have all your resources ready (including resources from the Resource Pack), have notes written up on the chalkboard, and are fully prepared to begin.
   - Classroom management also suggests that you plan which textbook activities and exercises will be done at which point in the lesson, and that you work through all exercises prior to the lesson.
   - In some cases, classroom management will also require you to photocopy an item for learners prior to the lesson, or to ensure that you have manipulatives such as boxes and tins available.

   The Learner Practice Table**.** This lists the relevant practice exercises that are available in each of the approved textbooks.

   - It is important to note that the textbooks deal with topics in different ways, and therefore provide a range of learner activities and exercises. Because of this, you will need to plan when you will get learners to do the textbook activities and exercises.
   - If you feel that the textbook used by your learners does not provide sufficient practice activities and exercises, you may need to consult other textbooks or references, including online references.
   - The *Siyavula* Open Source Mathematics textbooks are offered to anyone wishing to learn mathematics and can be accessed on the following website: https://www.everythingmaths.co.za/read

   C. ***Conceptual Development****:*

   This section provides support for the actual teaching stages of the lesson.

   Introduction: This gives a brief overview of the lesson and how to approach it. Wherever possible, make links to prior knowledge and to everyday contexts.

   Direct Instruction: Direct instruction forms the bulk of the lesson. This section describes the teaching steps that should be followed to ensure that learners develop conceptual understanding. It is important to note the following:

- Grey blocks talk directly to the teacher. These blocks include teaching tips or suggestions.
- Teaching is often done by working through an example on the chalkboard. These worked examples are always presented in a table. This table may include grey cells that are teaching notes. The teaching notes help the teacher to explain and demonstrate the working process to learners.
- As you work through the direct instruction section, and as you complete worked examples on the chalkboard, ensure that learners copy down:
  - formulae, reference notes and explanations
  - the worked examples, together with the learner's own annotations.
- These notes then become a reference for learners when completing examples on their own, or when preparing for examinations.
- At relevant points during the lesson, ensure that learners do some of the Learner Practice activities as outlined at the beginning of each lesson plan. Also, give learners additional practice exercises and questions from past papers as homework. Ensure that learners are fully aware of your expectations in this respect.

D. Additional Activities / Reading. This section provides you with web links related to the topic. Get into the habit of visiting these links as part of your lesson preparation. As teacher, it is always a good idea to be more informed than your learners. If possible, organise for learners to view video clips that you find particularly useful.

TRACKER

1. A Tracker is provided for each grade. The Trackers are CAPS compliant in terms of content and time.
2. You can use the Tracker to document your progress. This helps you to monitor your pacing and curriculum coverage. If you fall behind, make a plan to catch up.
3. Fill in the Tracker on a daily or weekly basis.
4. At the end of each week, try to reflect on your teaching progress. This can be done with the HoD, with a subject head, with a colleague, or on your own. Make meaningful notes about what went well and what didn't. Use the reflection section to reflect on your teaching, the learners' learning and to note anything you would do differently next time.
5. These notes can become an important part of your preparation in the following year.

RESOURCE PACK, ASSESSMENT AND POSTERS

1. A Resource Pack with printable resources has been provided for each term.
2. These resources are referenced in the lesson plans.
3. Two posters have been provided as part of the FET Mathematics Learning Programme for Term 1.
4. Ensure that the posters are displayed in the classroom.
5. Try to ensure that the posters are durable and long-lasting by laminating them, or by covering them in contact adhesive.
6. Note that you will only be given these resources once. It is important for you to manage and store these resources properly. You can do this by:
   - Writing your school's name on all resources
   - Sticking resource pages onto cardboard or paper
   - Laminating all resources, or covering them in contact paper
   - Filing the resource papers in plastic sleeves once you have completed a topic.
7. Add other resources to your resource file as you go along.
8. Note that these resources remain the property of the school to which they were issued..

ASSESSMENT AND MEMORANDUM

In the Resource Pack you are provided with assessment exemplars and memoranda as per CAPS requirements for the term.

CONCLUSION

Teacher support and development is a complex process. For successful Mathematics teachers, certain aspects of this Learning Programme may strengthen your teaching approach. For emerging Mathematics teachers, we hope that this Learning Programme offers you meaningful support as you develop improved structure and routine in your classroom, develop deeper conceptual understanding in your learners and increase curriculum coverage.

Term 1, Topic 1: Topic Overview

PATTERNS, SEQUENCES AND SERIES

A. TOPIC OVERVIEW

- This topic is the first of five topics in Term 1.
- This topic runs for three weeks (13,5 hours).
- It is presented over nine lessons.
- The lessons have been divided according to sub-topics, not according to one school lesson. An approximate time has been allocated to each lesson (which will total 13,5 hours). For example, one lesson in this topic could take two school lessons. Plan according to your school's timetable.
- Number Patterns counts 17% of the final Paper 1 examination.
- Mathematics is especially useful when it helps you predict, and number patterns are all about prediction
- Working with number patterns leads directly to the concept of functions in mathematics: a formal description of the relationships among different quantities.
- Recognising number patterns is also an important problem-solving skill. If a pattern is recognised when looked at systematically, the pattern can be used to generalise what can be seen in a broader solution to a problem.

Breakdown of topic into 8 lessons:

	Lesson title	**Suggested time (hours)**		**Lesson title**	**Suggested time (hours)**
1	Revision	1	6	Sigma Notation	1
2	Arithmetic sequences	1	7	Sum to infinity	2
3	Geometric sequences	2	8	Revision and Consolidation	1,5
4	Sum of Arithmetic sequences	2	9	Assignment	1
5	Sum of Geometric sequences	2			

B SEQUENTIAL TABLE

GRADE 11 and earlier	GRADE 12
LOOKING BACK	**LOOKING FORWARD**
● Linear patterns ● Quadratic patterns	● Arithmetic and Geometric sequences ● Arithmetic and geometric series, including sum to infinity ● Sigma notation ● Derivation of formulae for the sum of arithmetic and geometric series ● Problem solving

C WHAT THE NSC DIAGNOSTIC REPORTS TELL US

According to **NSC Diagnostic Reports** there are a number of issues pertaining to Patterns, Sequences and Series.

These include:

- Learners not always knowing the difference between the position and the value of the term
- Sigma notation not being interpreted correctly
- Learners need to analyse the type of sequence they are working with
- Algebraic skills (finding the difference between terms when variables are involved)

It is important that you keep these issues in mind when teaching this section.

When teaching Patterns, Sequences and Series encourage learners to look for patterns as well as patterns within patterns.

ASSESSMENT OF THE TOPIC

D

- CAPS formal assessment requirements for Term 1:
  - Investigation/Project
  - Assignment/ Test
  - Test
- Two tests, each with a memorandum, are provided in the Resource Pack. The tests are aligned to CAPS in every respect, including the four cognitive levels as required by CAPS (page 53). The tests are Resources 12 and 14 and the memoranda are Resources 13 and 15 in the Resource Pack.
- An assignment on Sequences and Series is provided in Resource Pack (Resource 1). The memorandum is provided in the lesson plan.
- The questions usually take the form of finding the general term of a given sequence then being asked further questions regarding the sequence as well as finding the sum of terms where possible.
- Monitor each learner's progress to assess (informally) their grasp of the concepts. This information can form the basis of feedback to the learners and will provide you valuable information regarding support and interventions required.

MATHEMATICAL VOCABULARY

E

Be sure to teach the following vocabulary at the appropriate place in the topic:

Term	Explanation
number pattern	List of numbers that follow a sequence or a pattern
consecutive	One after the other
common (or constant) difference	The value added each time to form a linear pattern
quadratic pattern	A sequence of numbers in which the second differences between each consecutive term differ by the same amount, called a common second difference

second difference	The second line of differences found in a number pattern The first line of differences forms a linear pattern and therefore the second line of differences is constant
arithmetic sequence	A sequence made by adding the same value each time Also known as a linear pattern.
arithmetic series	The sum of the terms of an arithmetic sequence
geometric sequence	A sequence of numbers where each term after the first is found by multiplying the previous one by a fixed, non-zero number called the common ratio
geometric series	The sum of the terms of a geometric sequence
constant ratio	The number used to multiply one term to get to the next term in a geometric sequence If division occurs, reciprocate and turn it into a multiplication Example: $\div 5 = \times \frac{1}{5}$
converging series	A series is convergent if the sum approaches a particular value as more terms are added
diverging sequence/ series	A sequence that does not converge For example, the sequence 2, 4, 6, 8, 10 …diverges because its limit is infinity The limit of a convergent series is a real number
rule/nth term / general term	Algebraic explanation of how a pattern is formed
term	A number in a given sequence Example: In the sequence: 5 ; 0 ; –5 ; –10 , all four numbers represent terms and each one of them hold a certain position
position	The place in the sequence held by one of the terms Example: In the sequence: 2 ; 4 ; 6 ; 8…. 6 is in the third position
sigma notation	The compact form of representing a series The symbol, Σ, is used

TERM 1, TOPIC 1, LESSON 1

REVISION OF NUMBER PATTERNS

Suggested lesson duration: 1 hour

POLICY AND OUTCOMES

A

CAPS Page Number	40
Lesson Objectives By the end of the lesson, learners will have revised: ● linear patterns ● quadratic patterns.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson, have the first example ready.
5. If there isn't a revision exercise in the textbook that you use, either use the revision exercise at the end of a Grade 11 textbook or items from a Grade 11 test on number patterns.
6. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		CLASSROOM MATHS		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
Rev	1			Qu's	9	1.1 1.2	3 4	1.1	2		

C CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. This topic will be easier for learners if time is spent ensuring that learners' knowledge of linear patterns and quadratic patterns is sound.

DIRECT INSTRUCTION

1. Ask learners what types of number patterns they have done in previous years (linear patterns and quadratic patterns). Write these on the board.
2. Ask: *What are the key points regarding each of these patterns?*
   (A linear pattern has a common/constant difference and a quadratic pattern has a constant second difference). Write these on the board.
3. Ask: *What are the formulae used for finding the general terms for these patterns?*
   (Linear: $T_n = a + (n - 1)d$, where a is the first term of the pattern and d is the common difference;
   Quadratic: $T_n = an^2 + bn + c$ where $2a$= second constant difference;
   $3a + b = 1^{st}$ difference of first two terms; $a + b + c = 1^{st}$ term)
   Write these on the board.
4. Ask learners to write down the summary that has now been written on the board.
5. Do two fully worked examples, covering some of these types of questions that could be asked. Learners should write the examples in their books, making notes as they do so.

Example 1

Use the linear pattern, 18 ;15 ;12… to answer the following questions:

a) List the next three terms
b) Find the nth term
c) Find the 25th term in the sequence
d) In which position would –15 lie in the sequence?
e) Is –52 part of the sequence?

Teaching notes:

a) Remind learners that a question like this would just require counting.

b) Use the newer method taught in Grade 11.
Remind learners that this requires knowing/finding the first term (a) and the common difference (d).

c) Ask: *What algebraic skill will be used to find a certain term in the sequence if given position?*
(Substitution).

d) Ask: *What algebraic skill will be used to find the position of a given term?*
(Solving equations).

e) Ask learners if anyone has any ideas how to answer this question. Listen to their responses – they may have a perfectly sound way of finding the solution. Listen carefully to check whether learners have the conceptual knowledge.
After the discussion:
Ask: *what types of numbers can n possibly be?*
(Only natural numbers – position cannot be a negative number or a fraction).
Tell learners that this is how you will approach this question. You will solve for n. If the answer is a natural number, then –52 is part of the sequence. If n is not a natural number, then –52 is not part of the sequence.

Solutions:

a) 9 ;6 ;3

b) $a = 18 \quad d = -3$

$T_n = a + (n-1)d$

$T_n = 18 + (n-1)(-3)$

$T_n = 18 - 3n + 3$

$\therefore T_n = -3n + 21$

c) $T_n = -3n + 21$

$T_{25} = -3(25) + 21$

$T_{25} = -75 + 21$

$T_{25} = -54$ ∴ the 25th term of the sequence is –54

d) $T_n = -3n + 21$

$-15 = -3n + 21$

$3n = 21 + 15$

$3n = 36$

$n = 12$ ∴ –15 is in the 12th position

e) $T_n = -3n + 21$

$-52 = -3n + 21$

$3n = 21 + 52$

$3n = 73$

$n = \frac{73}{3}$ ∴ –52 is not in the sequence because $\frac{73}{3}$ is not a natural number

Example 2

Use the pattern,

2 ;7 ;16 ;29

to answer the following questions:

a) Find the n^th^ term

b) Find T_9

c) In which position does the term 232 lie?

Teaching notes:

a) Ask: *What do we need to do to find the general term of a quadratic sequence?*
(Find the first and second differences).

b) Ask: *What algebraic skill is required to find the value of a term when given the position?*
(Substitution).

c) Ask: *What algebraic skill is required to find the position of a term given?*
(Solving an equation).
When doing the solution with learners:
Once the equation has been set up, ask: *What type of equation is this?*
(Quadratic equation).
Ask: *What does this mean?*
(That there should be two solutions)
Ask: *Can a term be in two different positions?*
(No).
Say: *Let's solve the equation and discuss the solutions*

Solutions:

a)

$2a = 4$ $\therefore a = 2$	$3a + b = 5$ $3(2) + b = 5$ $b = -1$	$a + b + c = 2$ $2 - 1 + c = 2$ $c = 1$

$\therefore$ the general term of the above quadratic number pattern is: $T_n = 2n^2 - 1n + 1$

In simplest form: $T_n = 2n^2 - n + 1$

b) $T_9 = 2(9)^2 - 9 + 1 = 154$

c)
$$T_n = 2n^2 - n + 1$$
$$232 = 2n^2 - n + 1$$
$$0 = 2n^2 - n - 231$$
$$0 = (2n + 21)(n - 11)$$
$$n = \frac{-21}{2} \text{ or } n = 11$$

$\therefore$ 232 is in the 11th position (position cannot be negative or a fraction).

Note:
Discuss ‘position’ with learners. Point out that a position can only be positive and a natural number. It is not possible to have a term in a negative position or in a fractional position.
The diagnostic reports state that learners are often confused between a term and the position.
Spend some time on the concepts ‘term’ and ‘position’ and ask directed questions using any patterns just used in examples to ensure that learners understand the difference.
For example, ask:
In this pattern, what term is in the 3rd position?
In this pattern, what is the position of the term …?

6. Ask directed questions so that you can ascertain learners’ level of understanding. Ask learners if they have any questions.
7. Give learners an exercise to complete with a partner.
8. Walk around the classroom as learners do the exercise. Support learners where necessary.

ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=UuceRRQGk8E
(Linear sequences – nth term)

https://www.youtube.com/watch?v=FfCq7bGAFoY
(Quadratic sequences – n^{th} term)

TERM 1, TOPIC 1, LESSON 2

ARITHMETIC SEQUENCES

Suggested lesson duration: 1 hour

A POLICY AND OUTCOMES

CAPS Page Number	40
Lesson Objectives By the end of the lesson, learners should be able to: ● recognise an arithmetic sequence ● find the general term of an arithmetic sequence ● answer other questions based on the arithmetic sequence such as position of a term and a finding a term in a given position.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson write the three sequences from point 2 on the board.
5. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		CLASSROOM MATHS		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
1	4	1 2	5 6	1	12	1.3 1.4	6 9	1.2 1.3	4 7	1.1 1.2	4 8

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Learners have been working with linear patterns, and therefore arithmetic sequences, since the Senior Phase.
2. This lesson provides the opportunity to consolidate learners' understanding of an arithmetic sequence. This will provide a good basis for the new work, finding the sum of a sequence, in a later lesson.

DIRECT INSTRUCTION

1. Ask: *What is the key idea linked to an arithmetic sequence?*
   (It has a constant difference).
2. Write the following three sequences on the chalkboard:

   5 ;7 ;9 ;11;…

   –11 ; -8 ; –5 ; –2 ;…

   3 ; –5 ; –13 ; –21…

3. Ask learners to write the first term and common difference of each sequence.

5 ;7 ;9 ;11;… $a = 5$; $d = 2$	-11 ; -8 ; -5 ; -2 ;… $a = -11$; $d = 3$	3 ; -5 ; -13 ; -21… $a = 3$; $d = -8$

4. Discuss the common difference. Remind learners that the difference is found by subtracting term 1 from term 2.

   $$d = T_2 - T_1$$

   However, make it clear that this should also be the same as $T_3 - T_2$. This is an important concept to remember for any arithmetic sequence.

5. Ask learners to write the following in their books:

   For any arithmetic sequence: $T_3 - T_2 = T_2 - T_1$

   This can be shown as: $d = T_{k+1} - T_k$

6. Before asking learners to find the general term of each sequence, show learners how the 'formula' is derived that they use to find the general term.

7. Use the 1st sequence to demonstrate: 5 ; 7 ; 9 ; 11;… ($a = 5$; $d = 2$)

$T_1 = 5 = \boldsymbol{a}$

$T_2 = 7 = \boldsymbol{a + d}$ (5 + 2)

$T_3 = 9 = \boldsymbol{a + 2d}$ (5 + 2(2))

$T_4 = 11 = \boldsymbol{a + 3d}$ (5 + 3(2))

8. Point out where the coefficient of d comes from. It is always one less than the position of the term. In other words, term 3 could have also been written as: $T_3 = a + (3 - 1)d$.

9. Tell learners to write the sequence and these 4 statements in their books.

10. Show learners how this idea can lead to a general term:

$$T_n = a + \mathbf{(n - 1)}d$$

11. Ask learners to find the general terms of each of the three sequences above.

$T_n = a + (n - 1)d$ $T_n = 5 + (n - 1)(2)$ $T_n = 5 + 2n - 2$ $\therefore\ T_n = 2n + 3$	$T_n = a + (n - 1)d$ $T_n = -11 + (n - 1)(3)$ $T_n = -11 + 3n - 3$ $\therefore\ T_n = 3n - 14$	$T_n = a + (n - 1)d$ $T_n = 3 + (n - 1)(-8)$ $T_n = 3 - 8n + 8$ $\therefore\ T_n = -8n + 11$

12. Do some fully worked examples with learners.
Learners should copy the examples in their books, taking notes as they do so.

Example:	Teaching notes:
Find a and d if $T_5 = 45$ and $T_{33} = 24$	Say: *Note that we have been given two pieces of information and have been asked to find two values. This is always the case – in functions it works the same.* Ask: *What statements can be made from the information given?* (Position is given so statements can be made using the general formula. The actual terms can then be put into the statements). Once this has been done, point out that simultaneous equations now need to be solved.

Solution:

$T_5 = a + (5 – 1)d$

$T_5 = a + 4d$

$45 = a + 4d$

$T_{33} = a + (33 – 1)d$

$T_{33} = a + 32d$

$24 = a + 32d$

$$
\begin{aligned}
45 – 4d &= a \\
24 &= a + 32d \\
24 &= 45 – 4d + 32d \\
24 – 45 &= 28d \\
–21 &= 28d \\
-\frac{3}{4} &= d \\
45 – 4d &= a \\
45 - 4\left(-\frac{3}{4}\right) &= a \\
45 + 3 &= a \\
48 &= a
\end{aligned}
$$

Example:	Teaching notes:
$3x – 5$; $x – 5$; $2x – 14$; … are the first three terms of an arithmetic sequence. a) Calculate the value of x b) Determine the general term of the sequence c) Find the 50th term d) Which term in the sequence is equal to –62	Ask: *What do we know about any arithmetic sequence?* (They have a common difference). Say: *This information will be used to find x.* Once this has been found, and hence the sequence is known, learners should answer the other questions easily.

Solution:

a) $T_3 - T_2 = T_2 - T_1$

$2x - 14 - (x - 5) = x - 5 - (3x - 5)$

$2x - 14 - x + 5 = x - 5 - 3x + 5$

$x - 9 = -2x$

$3x = 9$

$x = 3$

b) $a = 3(3) - 5 = 4$

$T_2 = 3 - 5 = -2$

$d = -2 - 4 = -6$

c) $T_n = a + (n - 1)d$

$T_n = 4 + (n - 1)(-6)$

$T_n = 4 - 6n + 6$

$\therefore\ T_n = -6n + 10$

$\therefore\ T_{50} = -6(50) + 10$

$T_{50} = -290$

d) $T_n = -6n + 10$

$-62 = -6n + 10$

$-72 = -6n$

$n = 12 \quad \therefore\ -62$ is the 12th term

13. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

14. Give learners an exercise to complete on their own.

15. Walk around the classroom as learners do the exercise. Support learners where necessary.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.mathplanet.com/education/algebra-2/sequences-and-series/arithmetic-sequences-and-series

https://www.youtube.com/watch?v=X7RByzgjK6Q

TERM 1, TOPIC 1, LESSON 3

GEOMETRIC SEQUENCES

Suggested lesson duration: 2 hours

POLICY AND OUTCOMES

A

CAPS Page Number	40

Lesson Objectives

By the end of the lesson, learners will have revised:

- recognise a geometric sequence
- find the general term of an geometric sequence
- answer other questions based on the geometric sequence such as position of a term and a finding a term in a given position.

B

CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**CLASSROOM MATHS**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
4	9	3 4	8 9	2	16	1.3 1.4	6 9	1.4	11	1.4 1.5 1.6	14 15 18

C CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Although geometric patterns were covered briefly in the Senior Phase, this section will still be quite new to learners.
2. Finding the general term and working with this has not been covered before.

DIRECT INSTRUCTION

1. Ask: *What is a geometric sequence?*
   Write a geometric sequence on the board to help learners:

   3 ; 6 ; 12 ; 24…

2. The multiplying of one term to get to the next term is the key issue that should have been mentioned. Tell learners that this is the common ratio.
   In the example above, 2 is the common ratio.
3. Write a few more examples of geometric sequences on the board and ask learners what the first term (a) and common ratio (r) is.

4 ; 12 ; 36 ; …	$a = 4$; $r = 3$
1 ; 10 ; 100 ; …	$a = 1$; $r = 10$
1 ; –2 ; 4 ; –8 …	$a = 1$; $r = -2$

4. Point out the last sequence. Tell learners to notice that when each term is the opposite sign of the previous term, they should always expect the common ratio to be negative.
5. Discuss the common ratio. Remind learners that the difference is found by dividing term 2 by term 1. This is required due to division being the inverse operation to multiplication and the ratio is the number being multiplied.

$$r = \frac{T_2}{T_1}$$

6. However, make it clear that this should also be the same as $\frac{T_3}{T_2}$. This is an important concept to remember for any arithmetic sequence.
   Ask learners to write the following in their books:

   For any geometric sequence, $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

This can be shown as: $r = \frac{T_{k+1}}{T_k}$

7. Show learners how the general term of a geometric sequence is derived.

8. Use the first sequence to demonstrate: 4 ; 12 ; 36 ; … (a = 4 ; r = 3)

$T_1 = 4 = \boldsymbol{a}$
$T_2 = 12 = \boldsymbol{a} \times \mathbf{3}$ (4×3)
$T_3 = 36 = \boldsymbol{a} \times \mathbf{3^2}$ $(4 \times 3 \times 3)$
$T_4 = 108 = \boldsymbol{a} \times \mathbf{3^3}$ $(4 \times 3 \times 3 \times 3)$

9. Point out where the exponent of r comes from. The exponent of r is always one less than the position of the term. In other words, term 3 could have also been written as: T_3=a×3^{3-1}

10. Tell learners to write the sequence and these four statements in their books.

11. Show learners how this idea can lead to a general term:

$$T_n = a.r^{n-1}$$

12. Find the general term of each of the three examples from above with learners.

4 ;12 ;36 ;…	$a = 4$; $r = 3$	$T_n = a.r^{n-1}$ $T_n = 4.3^{n-1}$
1 ;10 ;100 ;…	$a = 1$; $r = 10$	$T_n = a.r^{n-1}$ $T_n = 1.10^{n-1}$ $\therefore\ T_n = 10^{n-1}$
1 ; -2 ;4 ; -8…	$a = 1$; $r = -2$	$T_n = a.r^{n-1}$ $T = 1.(-2)^{n-1}$ $\therefore T_n = (-2)^{n-1}$

13. Give learners three more geometric sequences to find the general term of by themselves:

2 ;8 ;32…	10 ;-20 ; 40…	50 ;100 ;200…
Solutions		
$T_n = 2.4^{n-1}$	$T_n = 10.(-2)^{n-1}$	$T_n = 50.2^{n-1}$

14. Show learners how the first general term can be simplified because the value of both a and d can be written as base 2 then the laws of exponents can be used to simplify further.

$$T_n = 2.4^{n-1}$$
$$T_n = 2.(2^2\ )^{n-1}$$
$$T_n = 2.2^{2n-2}$$
$$T_n = 2^{2n-1}$$

15. Write the following geometric sequence on the board:

$$80 ;40 ;20 ;10\ldots$$

16. Ask: *What is the common ratio of this sequence?*
    Most learners tend to answer: ÷2.
    Point out that the common ratio needs to be what one term is multiplied with to get the next term. Ask again (if learners didn't get it the first time).
    The common ratio is $\frac{1}{2}$

17. Remind learners that when they notice division, they need to treat it as they would when dividing fractions – change it to multiplication and reciprocate.

18. Remind learners of what was discussed earlier. The common ratio is found by dividing term 2 by term 1. In the case of this sequence that would be $\frac{40}{80} = \frac{1}{2}$

19. Work with the learners to find the general term of this sequence:

$$T_n = a.r^{n-1}$$

$$T_n = 80.\left(\frac{1}{2}\right)^{n-1}$$

20. Point out:
    - When the ratio is a fraction, rather put it in a bracket to prevent confusion.
    - The 80 cannot be simplified by the 2 in the denominator as the denominator is part of the fraction that is raised to an unknown exponent – there could be many 2s represented here.

21. Give learners three more geometric sequences to find. Learners should work on their own.

20 ;10 ;5…	30 ;6 ; $\frac{6}{5}$ …	5 ; $\frac{5}{2}$; $\frac{5}{4}$…
Solutions		
$T_n = 20.\left(\frac{1}{2}\right)^{n-1}$	$T_n = 30.\left(\frac{1}{5}\right)^{n-1}$	$T_n = 5.\left(\frac{1}{5}\right)^{n-1}$

22. Ask learners to find the 6th term of each sequence:

$T_n = 20.\left(\frac{1}{2}\right)^{n-1}$	$T_n = 30.\left(\frac{1}{5}\right)^{n-1}$	$T_n = 5.\left(\frac{1}{2}\right)^{n-1}$
$T_6 = 20.\left(\frac{1}{2}\right)^{6-1}$ $T_6 = 20.\left(\frac{1}{2}\right)^{5}$ $= \frac{5}{8}$	$T_6 = 30.\left(\frac{1}{5}\right)^{6-1}$ $T_6 = 30.\left(\frac{1}{5}\right)^{5}$ $= \frac{6}{625}$	$T_6 = 5.\left(\frac{1}{2}\right)^{6-1}$ $T_6 = 5.\left(\frac{1}{2}\right)^{5}$ $= \frac{5}{32}$

23. Do some fully worked examples with learners.
    Learners should copy them into their books, taking notes as they do so.

Example:	Teaching notes:
Which term of the sequence $12\ ;4\ ;\ \frac{4}{3}\ldots$ is equal to $\frac{4}{243}$?	Tell learners to note that we need to find the position of a term in a sequence so the value of n will need to be found. Ask: *How can we find this?* (First find the general term then put $\frac{4}{243}$ into T_n and solve for n). Once the value of n has been found, point out that it is impossible for the value of n to anything but a natural number. Position can never be a fraction or negative.

Solution:

$a = 12\ ;\ r = \frac{1}{3}\left(\frac{T_2}{T_1} = \frac{4}{12}\right)$

$$T_n = a.(r)^{n-1}$$

$$T_n = 12.\left(\frac{1}{3}\right)^{n-1}$$

$$\frac{4}{243} = 12.\left(\frac{1}{3}\right)^{n-1}$$

$$\frac{1}{729} = \left(\frac{1}{3}\right)^{n-1}$$

$$\frac{1}{3^6} = \left(\frac{1}{3}\right)^{n-1}$$

$$3^{-6} = (3^{-1})^{n-1}$$

$$3^{-6} = 3^{-n+1}$$

$$\therefore -6 = -n + 1$$

$$n = 7$$

$\frac{4}{243}$ is in the 7th position

Tell learners that at this stage, finding the position of a term in a geometric sequence should always lead to bases that can be equal and therefore the exponents are equal.

Example:	Teaching notes:
Determine the first 4 terms of a geometric sequence where the 2nd term is 10 and the 9th term is $\frac{5}{64}$.	Say: *Note that we have been given two pieces of information and have been asked to find two values. This is always the case – in functions it works the same.* Ask: *What statements can be made from the information given?* (Position is given so statements can be made using the general formula. The actual terms can then be put into the statements). Once this has been done, point out that now simultaneous equations need to be solved.

Solution:

$T_2 = 10$	and	$T_9 = \frac{5}{64}$
$T_n = a.(r)^{n-1}$		$T_n = a.(r)^{n-1}$
$T_2 = a.(r)^{2-1}$		$T_9 = a.(r)^{9-1}$
$10 = a.(r)^1$		$\frac{5}{64} = a.(r)^8$
$\therefore\ ar = 10$		
$a = \frac{10}{r}$		

$\frac{5}{64} = a.(r)^8$ $\frac{5}{64} = \frac{10}{r}.(r)^8$ $\frac{5}{64} = 10r^7$ $\frac{1}{128} = r^7$ $\sqrt[7]{\frac{1}{128}} = \sqrt[7]{r^7}$ $\therefore\ r = \frac{1}{2}$	$a = \frac{10}{r}$ $a = \frac{10}{\frac{1}{2}}$ $a = 20$

The geometric sequence: 20 ;10 ;5 ; $\frac{5}{2}$

TOPIC 1, LESSON 3: GEOMETRIC SEQUENCES

Example:	Teaching notes:
A geometric sequence is given: $x - 4 \,;\, x + 2 \,;\, 3x + 1$ a) Calculate the value of x b) Determine the sequence c) Find the 6th term.	Ask: *What do we know about any arithmetic sequence?* (They have a common ratio). Say: *Use this information to find x.* $\frac{T_2}{T_1} = \frac{T_3}{T_2}$ Once x has been found, and hence the sequence is known, learners should answer the other questions easily.

Solution:

a)

$$\frac{T_2}{T_1} = \frac{T_3}{T_2}$$

$$\frac{x + 2}{x - 4} = \frac{3x + 1}{x + 2}$$

$$(x + 2)(x + 2) = (x - 4)(3x + 1)$$

$$x^2 + 4x + 4 = 3x^2 - 11x - 4$$

$$0 = 2x^2 - 15x - 8$$

$$0 = (2x + 1)(x - 8)$$

$$\therefore x = \frac{1}{2} \quad \text{or} \quad x = 8$$

Note that there are two possible sequences where these values could work.

b)

$x = -\frac{1}{2}$	$x = 8$
$x - 4 \,;\, x + 2 \,;\, 3x + 1$	$x - 4 \,;\, x + 2 \,;\, 3x + 1$
$-\frac{1}{2} - 4 \,;\, -\frac{1}{2} + 2 \,;\, 3\left(-\frac{1}{2}\right) + 1$	$8 - 4 \,; 8 + 2 \,; 3(8) + 1$
$-\frac{9}{2} \,;\, \frac{3}{2} \,;\, \frac{1}{2}$	$4 \,;\, 10 \,;\, 25$

c)

$x = \left(-\frac{1}{2}\right)$	$x = 8$
$-\frac{9}{2} \,;\, \frac{3}{2} \,;\, -\frac{1}{2}$	$4 \,;\, 10 \,;\, 25$
$a = -\frac{9}{2} \,;\, r = -\frac{1}{3}$	$a = 4 \,;\, r = \frac{5}{2}$
$T_n = a.(r)^{n-1}$	$T_n = \mathrm{a}.(\mathrm{r})^{n-1}$
$T_n = -\frac{9}{2}.\left(-\frac{1}{3}\right)^{n-1}$	$T_n = 4.\left(\frac{5}{2}\right)^{n-1}$
$T_6 = -\frac{9}{2}.\left(-\frac{1}{3}\right)^{6-1}$	$T_6 = 4.\left(\frac{5}{2}\right)^{6-1}$
$T_6 = \frac{1}{54}$	$T_6 = \frac{3125}{8}$

24. Ask directed questions so that you can ascertain learners' level of understanding.
    Ask learners if they have any questions.
25. Give learners an exercise to complete on their own.
26. Walk around the classroom as learners do the exercise. Support learners where necessary.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=cDRkMVTM0SE

https://www.youtube.com/watch?v=3xbormMmuK4

TERM 1, TOPIC 1, LESSON 4

SUM OF ARITHMETIC SEQUENCES

Suggested lesson duration: 2 hours

POLICY AND OUTCOMES

A

CAPS Page Number	40
Lesson Objectives By the end of the lesson, learners should be able to: ● find the sum of an arithmetic sequence ● derive the formula for finding the sum of an arithmetic sequence.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**CLASSROOM MATHS**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
4	18	5	12	3*	22*	1.7 1a, b, e 2a, b, d, f, g 4a, d, e	19	1.5	14	1.8	29

* Note: Via Afrika – leave out the questions with sigma notation as they should be done in lesson 6.

C CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. This lesson first deals with using the formulae to find the sum of an arithmetic sequence.
2. Once learners have practiced this, they will be shown how the formulae were derived. This could be examined.

DIRECT INSTRUCTION

1. Write the following arithmetic sequence on the board:

   3 ;5 ;7 ;9….

2. Tell learners that they will be finding the sum of the terms in this sequence.

   3 + 5 + 7 + 9

3. When finding the sum of a number of terms in a sequence, it becomes known as a series. A sequence is an ordered list of numbers and a series is the sum of a list of numbers. Tell learners to write this in their books.
4. Say: *There are two formulae that can be used to find the sum of an arithmetic sequence. Write these in your books. Use labels to show what each variable represents.*

$S_n = \frac{n}{2}[2a + (n - 1)d]$ n-position of final term required a-first term d-common difference	$S_n = \frac{n}{2}(a + l)$ n-position of final term required a-first term l-last term

5. Refer to the sequence you started the lesson with. Tell learners you are going to use this sequence to find the sum of the first 12 terms.
   Learners should write the sequence in their books.

3 ; 5 ; 7 ; 9….

$$Sn = \frac{n}{2}[2a + (n-1)d]$$

$n = 12$; $a = 3$; $d = 2$

$$S_{12} = \frac{12}{2}[2(3) + (12-1)(2)]$$

$$S_{12} = 6(6 + 22)$$

$$S_{12} = 168$$

(The sum of the first 12 terms is 168)

6. Tell learners that although we could have used the 2nd formula, it would have required more work. First, the general term would need to be found which would then be used to find the 12th term. This would be the last term in the series and could be used in the 2nd formula.

7. Do an example with learners where they would be more likely to choose the 2nd formula. Tell learners to write it in their books.

Calculate the sum of the following finite series:

$$6 + 10 + 14 + \ldots. + 82$$

Ask: *What information do we have?*

(First term, common difference and last term).

Ask: *Do we have enough information to use one of the formulae?*

(No – the number of terms being added is required).

Ask: *How can we find the number of terms?*

(Find the general term then use this to find the position of 82).

Solution:

$$T_n = a + (n-1)d$$

$$T_n = 6 + (n - 1)(4)$$

$$T_n = 6 + 4n - 4$$

$$T_n = 4n + 2$$

$T_n = 4n + 2$

$82 = 4n + 2$

$80 = 4n$

$20 = n$

Say: *Although either of the two formulae could be used, the 2nd formula requires less work.*

$$S_n = \frac{n}{2}(a + l)$$

$$S_{20} = \frac{20}{2}(6 + 82)$$

$$S_{20} = 10(88)$$

$$S_{20} = 880$$

8. Do some fully worked examples with learners.
   Learners should write the examples in their books, taking notes as they do so.

<table>
<tr>
<td>Example:
Given the arithmetic series: 2 + 9 + 16 + ⋯ (to 251 terms).
a) Write down the 4th term of the series
b) Calculate the 251st term of the series
c) Calculate the sum of the series.
d) How many terms in the series are divisible by 4?
DBE NOV 2014</td>
<td>Teaching notes:
a) The 4th term should be easy to find using the common difference.
b) Learners should be able to see that they have the a,d and n and can therefore use the general term of an arithmetic sequence to find the 251st term.
c) Remind learners that they can use either of the two formulae.
d) Ask learners to write out more of the series so they can get an idea of where the multiples of 4 are. Once this has been done, tell learners to check if the last term is divisible by 4. They should then use this information to create a new series and find the position of the last term.
(Note: There are a number of ways this could be answered – question learners then choose the method most suited to the learners' answers).</td>
</tr>
<tr>
<td>a) $T_4 = 23$
b) $T_{251} = a + (n – 1)(d)$
$T_{251} = 2 + (251 – 1)(7)$
$T_{251} = 1752$</td>
<td>c) $S_n = \frac{n}{2}(a + l)$
$S_{251} = \frac{251}{2}(2 + 1752)$
$S_{251} = 220\ 127$</td>
</tr>
<tr>
<td colspan="2">d) Series: 2 + 9 + **16** + 23 + 30 + 37 + **44** … + **1752**
New series: 16 + 44 + ⋯ + 1752
$T_n = a + (n - 1)(d)$
$T_n = 16 + (n – 1)(28)$
$T_n = 16 + 28n – 28$
$T_n = 28n – 12$
$1752 = 28n – 12$
$1764 = 28n$
$n = 63$
There are 63 terms that are divisible by 4</td>
</tr>
</table>

<table>
<tr><td>Example 1
An arithmetic series has 21 terms. The first term is 3 and the last term is 53. Find the sum of the series.</td><td>Teaching notes:
Ask: *What information has been given?*
(The number of terms; the first term; the last term).
Say: *There is enough information to use one of the formulae.*</td></tr>
<tr><td colspan="2">Solution:
$n = 21$;$a = 3$; $T_{21} = 53$
$S_n = \frac{n}{2}(a + l)$
$S_{21} = \frac{21}{2}(3 + 53)$
$S_{21} = 588$</td></tr>
<tr><td>Example 2
A runner decides to start a new training programme by running 2km on the first day and increasing the distance by 1km every day for two weeks. Find the total distance run at the end of the two weeks.</td><td>Teaching notes:
Ask learners to write the first four terms of the sequence created from the distances.
Ask: *Is there enough information to find the sum of the series?*
(Yes – 14 terms; first term; common difference)</td></tr>
<tr><td colspan="2">Solution:
Series: 2 + 3 + 4…   Two weeks = 14 days
$S_n = \frac{n}{2}[2a + (n - 1)d]$
$n = 14$; $a = 2$; $d = 1$
$S_{14} = \frac{14}{2}[2(2) + (14 - 1)(1)]$
$S_{14} = 7[4+13]$
$S_{14} = 119$
The total distance run in two weeks was 119km.</td></tr>
</table>

9. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

10. Give learners an exercise to complete on their own.

11. Walk around the classroom as learners do the exercise. Support learners where necessary.

12. Once the exercise has been completed and corrected, tell learners you are going to show them how the formulae that they have been using were derived.

Explain to learners that because we want to derive the formula we need to 'pretend' we don't know it yet. Tell them that you are going to start by using a numerical example to assist them in understanding better when you use variables. Ensure learners write everything in their books.

13. Find S_5 for the series, 2 + 5 + 8 …

    Point out that we could easily list this series:

    $$S_5 = 2 + 5 + 8 + 11 + 14$$

    Write the series in reverse underneath this one.

    $$S_5 = 2 + 5 + 8 + 11 + 14$$

    $$S_5 = 14 + 11 + 8 + 5 + 2$$

    Add these two series in columns:

    $$\begin{aligned} S_5 &= 2 + 5 + 8 + 11 + 14 \\ \underline{S_5} &\underline{= 14 + 11 + 8 + 5 + 2} \\ 2S_5 &= 16 + 16 + 16 + 16 + 16 \\ 2S_5 &= 5 \times 16 \\ S_5 &= \frac{5}{2} \times 16 \end{aligned}$$

14. Ask: *What is the connection with the sequence and 16?*

    (16 is the sum of the first and last term).

15. Say: *Now we need to use the same idea but with general terms and not actual values.*

16. Write the following on the board. As you write, confirm that learners understand why each term is written as it is:

$$S_n = \underset{1^{st}}{a} + \underset{2^{nd}}{(a+d)} + \underset{3^{rd}}{(a+2d)} + \ldots + \underset{\text{last}}{[a+(n-1)d]}$$

Say: The method we used at the beginning will not work with this example unless we have the same number of terms on each side of the 'gap'.
Ask: What will the 2nd last term and 3rd last term be called?
[a+(n - 2)d] and [a+(n - 3)d]
Update the statement.

$$S_n = a + (a+d) + (a+2d) + \ldots + [a+(n-3)d] + [a+(n-2)d] + [a+(n-1)d]$$

Confirm that learners understand to this point before continuing.
Say: *We need to reverse this series and write it underneath the one we already have then add the two statements together, collecting like terms.*

$$\begin{aligned} S_n &= a + (a+d) + (a+2d) + \ldots + [a+(n-3)d] + [a+(n-2)d] + [a+(n-1)d] \\ S_n &= [a+(n-1)d] + [a+(n-2)d] + [a+(n-3)d] + (a+2d) + (a+d) + a \\ \hline 2S_n &= [2a+(n-1)d] + [2a+(n-1)d] \ldots + n[2a+(n-1)d] \ldots + [2a+(n-1)d] + [2a+(n-1)d] + [2a+(n-1)d] \end{aligned}$$

You may need to do each addition on the side with learners.

For example, in column 2:

$(a+d)+[a+(n-2)d]$

$= a+d+a+dn-2d$

$= 2a+dn-d$

$= 2a+d(n-1)$

$= 2a+(n-1)d$

Ask: *How many terms have been added?*

(n terms. Ensure that learners understand that it is not 6 terms).

Remind learners how the example with actual values was shortened when there were many of the same term. The addition was changed into a multiplication using the number of terms.

$$2S_n = [2a+(n-1)d]+[2a+(n-1)d]+n[2a+(n-1)d]\ldots+[2a+(n-1)d]+\ [2a+(n-1)d]+[2a+(n-1)d]$$

$$2S_n = n\times[2a+(n-1)d]$$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

Discuss the expression in the bracket with learners:

$[2a+(n-1)d]$

Expand: $a+a+(n-1)d$

Underline all but the first a. Ask: *Which term in the sequence is represented?*

$a+\mathbf{a+(n-1)d}: a+(n-1)d$ represents the last term.

$a+a+(n-1)d$ can therefore be rewritten as: $a+l$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$S_n = \frac{n}{2}[a+a+(n-1)d]$$

$$S_n = \frac{n}{2}[a+l]$$

17. Tell learners that in the next lesson they will be finding the sum of geometric sequences.

ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=Dj1JZIdIwwo

https://www.youtube.com/watch?v=8L6oRtje8xw

TERM 1, TOPIC 1, LESSON 5

SUM OF GEOMETRIC SEQUENCES

Suggested lesson duration: 2 hours

A POLICY AND OUTCOMES

CAPS Page Number	40
Lesson Objectives By the end of the lesson, learners should be able to: ● find the sum of a geometric sequence ● derive the formula for finding the sum of a geometric sequence.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**CLASSROOM MATHS**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
5	23	6	14	4*	25*	1.7 1c, d, f 2b, e 4b, c	19	1.6	19	1.9	35

* Note: Via Afrika – leave out the questions with sigma notation as they should be done in lesson 6

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. This lesson first deals with using the formulae to find the sum of a geometric sequence.
2. Once learners have practiced this, they will be shown how the formulae were derived. This could be examined.

DIRECT INSTRUCTION

1. Write the following geometric sequence on the board:

   3 ; 6 ; 12 ; 24….

2. Tell learners that they will be finding the sum of the terms in this sequence:

   3 + 6 + 12 + 24

3. Remind learners that a sequence is an ordered list of numbers and a series is the sum of a list of numbers.
4. Say: *There is only one formula for finding the sum of a geometric sequence. There are, however, two versions of it (this will be explained at the end of the lesson when the formula is derived). Both work – there is no real difference between them. Write these in your books with labels for what each variable represents.*

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

n – position of final term required
a – first term
r – common ratio

5. Refer to the sequence you started the lesson with. Tell learners you are going to use this sequence to find the sum of the first 8 terms. Learners should write it in their books.

   3 ; 6 ; 12 ; 24 ….

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$n = 8$; $a = 3$; $r = 2$

$$S_8 = \frac{3(2^8 - 1)}{2 - 1}$$

$$S_8 = \frac{3(255)}{1}$$

$$S_8 = 765$$

(The sum of the first 8 terms is 765)

6. Tell learners to use the other version of the formula to confirm that they still get 765 as the solution.
7. Do some fully worked examples with learners.
   Learners should write them in their books, taking notes as they do so.

Example 1	Teaching notes:
The first three terms of a sequence are: 16 ; 4 ; 1 a) Calculate the value of the 12th term (leave your answer in simplified exponential form) b) Calculate the sum of the first 10 terms of the sequence. NSC NOV 2014	a) Learners did this in a previous lesson. Ask: *What will you need to use to find the 12th term?* (The general term of a geometric sequence). b) Confirm that learners have all the variables required to use the sum of a geometric sequence formula.

Solution:

a) $T_n = a.r^{n-1}$

$$T_{12} = 16.\left(\frac{1}{4}\right)^{12-1}$$

$$T_{12} = 4^2.(4^{-1})^{11}$$

$$T_{12} = 4^{-9}$$

b) $S_n = \frac{a(r^n - 1)}{r - 1}$

$$S_{10} = \frac{16\left(\left(\frac{1}{4}\right)^{10} - 1\right)}{\frac{1}{4} - 1}$$

$$S_{10} = 21{,}33$$

Example 2	Teaching notes:
The number of members of a new social networking site doubles every day. a) If there are 120 members on day 1, how many members will there be on day 7? b) The site earns 5c per member per day. How much will they have earned by the 10th day? Give your answer in rands.	a) This question is similar to the first part of the previous question. b) Learners need to notice that this is a sum of a geometric sequence question.

Solution:

$$120 \, ; \, 240 \, ; \, 480 \, ..$$

$$a = 120 \, ; \, r = 2$$

a) $T_n = a.^{n-1}$

$T_n = 120.2^{n \ 1}$

$T_7 = 120.2^{7-1}$

$T_7 = 7680$

b) $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$S_{10} = \dfrac{120(2^{10} - 1)}{2 - 1}$

$S_{10} = 122760$

The site will have earned R1227,60

8. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

9. Give learners an exercise to complete on their own.

10. Walk around the classroom as learners do the exercise. Support learners where necessary.

11. Once the exercise has been completed and corrected, tell learners you are going to show them how the formulae that they have been using were derived.

> Explain that because we want to derive the formula we need to 'pretend' we don't know it yet. Tell them that you are going to start by using a numerical example to assist them in understanding better when you use variables. Ensure learners write everything in their books.

12. Find S_5 for the series, 6 + 18 + 54 …

    Point out that we could easily list this series:

$$S_5 = 6 + 18 + 54 + 162 + 486$$

Tell learners that now you're going to multiply the series by the common ratio (3) and write the new series underneath this one:

$$\begin{aligned} S_5 &= 6 + 18 + 54 + 162 + 486 \\ 3S_5 &= 18 + 54 + 162 + 486 + 1458 \end{aligned}$$

Subtract: S_5 - $3S_5$. Point out that moving the first term of the new sequence underneath the second term of the original sequence will make this quicker to subtract.

$$\begin{aligned} S_5 &= 6 + 18 + 54 + 162 + 486 \\ -(3S_5 &= \quad\;\; 18 + 54 + 162 + 486 + 1458) \\ \hline -2S_5 &= 6 + 0 + 0 + \;\; 0 + \;\; 0 - 1458 \\ -2S_5 &= -1452 \\ S_5 &= 726 \end{aligned}$$

13. Say: *Now we need to use the same idea but with general terms and not actual values.*

14. Write the following on the board. As you write, confirm that learners understand why each term is written as it is:

$$S_n = \underset{1^{\text{st}}}{a} + \underset{2^{\text{nd}}}{ar} + \underset{3^{\text{rd}}}{ar^2} + \ldots + \underset{\text{last}}{ar^{n-1}}$$

Ask: *What is the common ratio?*

(r).

Say: *Multiply the first statement by r and write the new series underneath the original series. Subtract.*

$$\begin{aligned} S_n &= a + ar + ar^2 + \ldots + ar^{n-1} \\ -(rS_n &= \quad\;\; ar + ar^2 + \ldots + ar^{n-1} + ar^n) \\ \hline S_n - rS_n &= a + 0 + 0 + \;..\; + \;\; 0 \;\; - \; ar^n \\ S_n - rS_n &= a - ar^n \\ S_n(1 - r) &= a(1 - r^n) \\ S_n &= \frac{a(1 - r^n)}{1 - r} \end{aligned}$$

Point out that if we had subtracted S_n from rS_n, we would have derived the other version of the formula:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

15. Ask learners if they have any questions.

ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=mYg5gKlJjHc

https://www.youtube.com/watch?v=mcnblnEsf98

TERM 1, TOPIC 1, LESSON 6

SIGMA NOTATION

Suggested lesson duration: 1 hour

A POLICY AND OUTCOMES

CAPS Page Number	40

Lesson Objectives

By the end of the lesson, learners should be able to:

- find the sum of a series when given the sigma notation
- write a series in sigma notation.

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson have the two series from point 1 and the sigma symbol ready.
5. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**CLASSROOM MATHS**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
3	13	8	18	3 4	22 25	1.6 1.7 (3)	14 19	1.7	23	1.7	23

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Clarify that sigma notation is a short-hand form of writing a series of terms.
2. Learners need to understand what is being depicted when they read sigma notation and must also be able to write a series into sigma notation.

DIRECT INSTRUCTION

1. Write the following two series on the board:

3 + 5 + 7 + 9	512 + 256 + 128 + 64

2. Say: *Today we are going to learn a short-hand way to write a series similar to the two series above.*
3. Put the sigma symbol on the board:

$$\sum$$

4. Say: *This is a letter from the Greek alphabet. It is called Sigma. In mathematics, it means to 'sum up'. This symbol will be used to show that it is required to find the sum of a series.*
5. On the symbol, write the three main requirements – the rule/formula; the starting term and the ending term. Tell learners to copy it into their books.

Last value of n

$$\sum_{n=1}^{6} 2n$$

Formula for each term

First value of n

6. Discuss each label with learners:

First value of n	This tells us which term to start with in the sequence It may not always be the first term
Last value of n	This tells us which term to end at for the sum of terms

Formula/General term	This gives us the general term to be used for each of the terms to be added

7. Tell learners when they are given a series; they can assume that the first term given is the first term of the series.

8. Ask learners to write the two series from the beginning of the lesson in sigma notation. Once learners have had a chance to do so, do each one with them.

	3 + 5 + 7 + 9	512 + 256 + 128
Step 1 First value of n	$n = 1$	$n = 1$
Step 2 Last value of n	$n = 4$	$n = 3$
Step 3 General term	$T_n = a + (n - 1)(d)$ $T_n = 3 + (n - 1)(2)$ $T_n = 3 + 2n - 2$ $T_n = 2n + 1$	$T_n = a.r^{n-1}$ $T_n = 512.\left(\frac{1}{2}\right)^{n-1}$
Sigma Notation	$\sum_{n=1}^{4}(2n+1)$	$\sum_{n=1}^{3} 512\left(\frac{1}{2}\right)^{n-1}$

9. Tell learners to find the sum of the two series:

$\sum_{n=1}^{4}(2n+1)$ $= 3 + 5 + 7 + 9 = 24$	$\sum_{n=1}^{3} 512\left(\frac{1}{2}\right)^{n-1}$ $= 512 + 256 + 128 = 896$

10. Give learners the following examples to try on their own. They should set out the examples as above.
While learners are working, walk around the classroom to assist where necessary.

$\sum_{k=1}^{3}(2k+1)$	$\sum_{k=0}^{4}(-2)^k$	$\sum_{k=2}^{5}\left(3k^2-7\right)$

11. Once learners have completed the three questions, do them on the board for learners to correct their work.

$$\sum_{k=1}^{3}(2k+1) = (2\times1+1)+(2\times2+1)+(2\times3+1)$$
$$= 3+5+7$$
$$= 15$$

$$\begin{aligned}\sum_{k=0}^{4}(-2)^k &= (-2)^0+(-2)^2+(-2)^3+(-2)^4\\ &= 1+(-2)+4+(-8)+16\\ &= 11\end{aligned}$$

$$\begin{aligned}\sum_{k=2}^{5}\left(3k^2-7\right) &= \left(3\times 2^2-7\right)+\left(3\times 3^2-7\right)+\left(3\times 4^2-7\right)+\left(3\times 5^2-7\right)\\ &= 5+20+41+68\\ &= 134\end{aligned}$$

12. Tell learners that you are going to do one more example with them from a past paper. They should write it in their books and make notes as they do so.

Example:	Teaching notes:
Given the geometric sequence: $-\frac{1}{4}$; b; –1 a) Calculate the possible values of b If $b = \frac{1}{2}$: b) find T_{19} c) write the sum of the first 20 positive terms of the sequence in sigma notation NSC FEB 2017	a) Ask: *What is the key concept of geometric sequences?* (They have a common ratio which means $\frac{T_2}{T_1}=\frac{T_3}{T_2}$). Say: *This will be used to find the value of b* b) Say: *When given the sequence so the value of a and r is known, the general term can be used to find a term in a given position.* Tell learners they should first write the original sequence and extend it by a few terms; then they can write down the new sequence using only the positive terms. Once they have the sequence, they should know a and r and therefore be able to find the three components required for sigma notation.

Solution:

a) $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$$\frac{b}{-\frac{1}{4}} = \frac{-1}{b}$$

$$b^2 = \frac{1}{4}$$

$$\therefore\ b = \pm\frac{1}{2}$$

b) $\frac{1}{4}\ ;\ \frac{1}{2}\ ;\ -1$

$$a = \frac{1}{4}\ ;\ r = -2$$

$$T_n = ar^{n-1}$$

$$T_{19} = -\frac{1}{4}(-2)^{19-1}$$

$$T_{19} = -2^{16} = -65536$$

c) Sequence: $\frac{1}{4}\ ;\ \frac{1}{2}\ ;\ -1\ ;\ 2\ ;\ -4\ ;\ 8\ \ldots$

Positive terms only: $\frac{1}{2}\ ;\ 2\ ;\ 8\ \ldots$

$$a = \frac{1}{2}\ ;\ r = 4$$

$$\sum_{n=1}^{20}\left(\tfrac{1}{2}\right)(4)^{n-1}$$

13. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.
14. Give learners an exercise to complete on their own.
15. Walk around the classroom as learners do the exercise. Support learners where necessary.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=H6rVcFCxU3g

TERM 1, TOPIC 1, LESSON 7

SUM TO INFINITY

Suggested lesson duration: 2 hours

POLICY AND OUTCOMES

A

CAPS Page Number	40
Lesson Objectives By the end of the lesson, learners should be able to ● explain the difference between a diverging and converging geometric series ● find the sum to infinity of a converging geometric series.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson have the two series from point 2 ready.
5. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**CLASSROOM MATHS**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
7	33	7	15	5	29	1.10 1.11	26 36	1.8 1.9	24 31	1.10 1.11	38 42

C CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Learners have learned about geometric series and how to find the sum of a finite number of terms.
2. The new concept of adding an infinite number of terms is explored in this lesson.

DIRECT INSTRUCTION

1. Say: *In this lesson we are going to consider two different types of geometric series.*
2. Write the following series on the board:

$2 + 1 + \frac{1}{2} + \frac{1}{4} \ldots$	$3 + 6 + 12 + 24 \ldots$

3. Say: *Find the common ratio for each series:*

$2 + 1 + \frac{1}{2} + \frac{1}{4} \ldots$ $r = \frac{1}{2}$	$3 + 6 + 12 + 24 \ldots$ $r = 2$

4. Ask: *How would you find the sum of a certain number of terms in each of these series?*

   (By using the formula $S_n = \frac{a(r^n - 1)}{r - 1}$).

5. Point out that the reason they are able to do this with ease is because they would be told exactly how many terms are required.
   Ask: *Is it possible to find the sum of terms in the entire sequence?*
   (Learners would more than likely say no).
   Ask: *Why not?*
   (It seems impossible to add a list of numbers that keep getting bigger or smaller. We wouldn't know when to stop).

6. Tell learners that you are going to investigate this idea with them.
7. Refer learners to the first sequence that was written down:

$$2 + 1 + \frac{1}{2} + \frac{1}{4} \ldots$$

8. Ask learners to find the general term in its simplest form:

$$T_n = a.r^{n-1}$$
$$a = 2 ; r = \frac{1}{2}$$
$$T_n = 2.\left(\frac{1}{2}\right)^{n-1}$$
$$T_n = 2.(2^{-1})^{n-1}$$
$$T_n = 2.2^{-n+1}$$
$$T_n = 2^{2-n}$$

9. Ask learners to find each of the following one at a time: (put one up, let learners find the answer then put the next one up). Tell learners not to use the Sum formula, but to rather keep adding a term to their previous answer.

$2 + 1 + \frac{1}{2} + \frac{1}{4} \ldots$		
	For reference	Solution
S_1	2	2
S_2	$2 + 1$	3
S_3	$2 + 1 + \frac{1}{2}$	$3\frac{1}{2}$
S_4	$2 + 1 + \frac{1}{2} + \frac{1}{4}$	$3\frac{3}{4}$
S_5	$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$	$3\frac{7}{8}$
S_6	$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$	$3\frac{15}{16}$

10. Stop and ask if anyone notices something interesting. Once you have a few responses, Ask: *What do you think is going to happen if we add a few more terms?*

If the response is that the answer will keep getting closer to 4 and not exceed it, you can move on.
If learners don't notice that there will be a limit of 4, continue with the table above, then ask the question again.

	For reference	Solution
S_6	$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$	$3\frac{31}{32}$
S_7	$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$	$3\frac{63}{64}$
S_8	$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$	$3\frac{127}{128}$
S_9	$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256}$	$3\frac{255}{256}$

11. Ensure that learners notice the following trend: As the number of terms being added increases (n), the sum of the terms increases (S_n) but by smaller and smaller amounts.

12. In the example above, the sum of all the terms is getting closer and closer to 4.

> Tell learners that when they learn Calculus next term, the idea of this limit will be explored further.

13. Refer learners back to the two series written on the board at the beginning of the lesson. Ask them to note the difference between the two – for the first series, it seems we can keep adding the next term (better known as the sum to infinity) and know that the sum will get closer and closer to 4 but will never surpass it.
    This is called a converging series because the sum goes towards (converges) a particular value.

14. The second series just gets larger and larger and there doesn't seem to be a way to add all the terms for ever and ever.
    This is called a diverging series because the sum increases indefinitely.

15. Ask learners to note the common ratio in each of the two series. Point out that the common ratio in the first series, in which the (converging) series could be found indefinitely is $\frac{1}{2}$ whereas the common ratio for the second series, in which the (diverging) series could not be found is 2.

16. Summarise this important distinction for learners and tell learners to write it in their books:

Common ratio:	Therefore, the series is:	Can we find the sum to infinity?
$-1 < r < 1$	converging	Yes
$r \leq -1$ or $r \geq 1$	diverging	No

17. Say: *The sum to infinity (an endless list) can ONLY be found if the common ratio is between* –1 *and* 1.

18. Tell learners that there is a formula to find the sum to infinity.
    Write the formula on the board:

$$S_\infty = \frac{a}{1-r}$$

19. Use this formula to show learners what the sum to infinity would be for the series that has been used throughout the lesson.

<table>
<tr>
<td>Example
Find the sum to infinity (S_∞) of
$2 + 1 + \frac{1}{2} + \frac{1}{4} \ldots$</td>
<td>Tell learners they need to find a and r.
Remind learners to confirm that the ratio lies between -1 and 1!
Use the formula.</td>
</tr>
<tr>
<td colspan="2">Solution:
$a = 2 \,;\, r = \frac{1}{2}$
$S_\infty = \frac{a}{1 - r}$
$S_\infty = \frac{2}{1 - \frac{1}{2}}$
$S_\infty = \frac{2}{\frac{1}{2}}$
$S_\infty = 4$</td>
</tr>
</table>

20. Remind learners what the limit was when they were adding one term at a time.

21. Ask if anyone has any questions before doing further examples.

22. Do two fully worked examples with learners.
    Learners should write the example in their books, making notes as they do so.

Example	Teaching notes:
Given the geometric series: $(x - 2) + (x^2 - 4) + (x^3 + 2x^2 - 4x - 8)$ .. a) Determine the value of x for which the series converges b) If $x = -\frac{3}{2}$, calculate the sum to infinity of the given series.	a) Ask: *What can you tell me about a converging series?* (The ratio must lie between –1 and 1). Say: *We need to find the ratio in terms of x, then substitute this into the inequality* –1 < r < 1 and solve for x. b) Tell learners to first write down the series using $x = -\frac{3}{2}$. Once this has been done, finding the sum to infinity should be straightforward.

Solution:

a) $r = \frac{(x^2-4)}{(x-2)}$

$r = \frac{(x+2)(x-2)}{x-2}$

$r = x + 2$

$-1 < r < 1$

$-1 < x + 2 < 1$

$-3 < x < -1$

b) If $x = -\frac{3}{2}$: $\left(-\frac{3}{2}-2\right)+\left(\left(-\frac{3}{2}\right)^2-4\right)+\left(\left(-\frac{3}{2}\right)^3+2\left(-\frac{3}{2}\right)^2-4\left(-\frac{3}{2}\right)-8\right)..$

$\left(-\frac{7}{2}\right)+\left(-\frac{7}{4}\right)+\left(-\frac{7}{8}\right)..$

$a = -\frac{7}{2}$; $r = \frac{1}{2}$

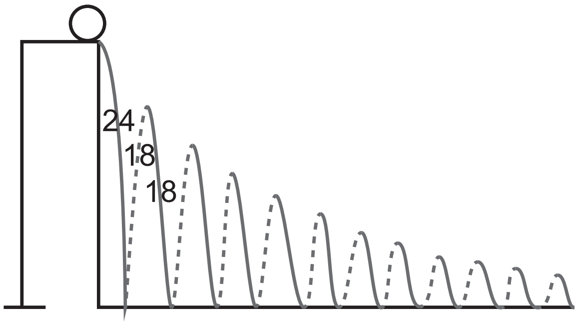
$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{-\frac{7}{2}}{1-\frac{1}{2}}$$

$$S_\infty = -7$$

Example:	Teaching notes:
A ball is dropped from a small table that is 24cm high. The ball always rebounds three fourths of the distance fallen. Approximately how far will the ball have traveled when it finally comes to rest?	It is important that you draw a diagram on the board to demonstrate the situation. Learners need to be clear that they will need to consider the rebounding part of the ball. In other words, the series they need to consider is 24 (falling) + 18 (rebounding) + 18 (falling) +13,5 (rebounding) and so on. Point out that each distance is doubled except the first fall of 24cm. Therefore, 24 can be dealt with on its own and from 18 onwards can be considered the sequence that is required to find the sum to infinity of. This must be multiplied by 2.

Solution:



Series: 24 + 2(18 + 13,5 + 10,125 …)

$$18 + 13{,}5 + 10{,}125 \ldots$$

$$a = 18 \; ; r = \frac{3}{4}$$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{18}{1-\frac{3}{4}}$$

$$S_\infty = 72$$

∴ distance travelled: 24 + 2(72) = 168cm

23. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

24. Give learners an exercise to complete on their own.

25. Walk around the classroom as learners do the exercise. Support learners where necessary.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=MTOKAA8rRA0

https://www.youtube.com/watch?v=Soi8GQdPBmg

TERM 1, TOPIC 1, LESSON 8

REVISION AND CONSOLIDATION

Suggested lesson duration: 1,5 hours

A POLICY AND OUTCOMES

CAPS Page Number	40
Lesson Objectives By the end of the lesson, learners will have revised: ● all the concepts covered in sequences and series.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson have the first few questions ready.
5. The table below provides references to this topic in Grade 12 textbooks.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		CLASSROOM MATHS		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
6 Rev Some Ch	27 35 36	9 10 Rev	21 25 27	Qu's	34	1.8 Rev	22 38	1.10 1.11	33 35	1.12	44

MIND ACTION SERIES		PLATINUM		VIA AFRIKA	
EX	PG	EX	PG	EX	PG
Invest	37	Assign	29	Project	34

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Ask learners to summarise what they have learned in this section. Point out issues that you know are important as well as problems that you encountered from your own learners.
2. If learners want you to explain a concept again, do that now.

DIRECT INSTRUCTION

This lesson is made up of fully worked examples from a past examination covering most of the concepts in this topic. As you work through these examples with the learners, it is important to frequently talk about as many concepts as possible.
For example, use the words sequence, arithmetic, quadratic, geometric, sum, series and general term.

Say: *I am going to do an entire Sequences and Series question from the 2015 final examination with you. You should write them down as I do them, taking notes at the same time.*

Questions

The following sequence is given: 10 ; 5 ; 2,5 ; 1,25 …

1. Calculate the value of the 5th term, T_5, of this sequence.
2. Determine the nth term T_n, in terms of n.
3. Explain why the infinite series 10 + 5 + 2,5 + 1,25 + … converges.
4. Determine $S_\infty - S_n$ in the form ab^n, where S_n is the sum of the first n terms of the sequence.

Teaching notes

1. Learners first need to recognise that this is a geometric sequence. Remind learners to look for a common difference, then consider a second common difference and finally divide T_2 by T_1 and T_3 by T_2 to check for a common ratio.
   As there are already four terms and only the 5th one has been asked, learners can multiply T_4 by the common ratio to get T_5.
2. Learners need to know the first term and the common ratio to find the general term. These should have been established in the previous question.
3. Ask: *What is a converging series?*
   (A series is convergent if the sum approaches a certain value as more terms are added).
   Ask: *For a geometric series to converge, what is required?*
   (The ratio must lie between –1 and 1).
4. Tell learners that this may seem like an unusual question, but they should approach it by thinking of each part first.
   Ask: *How do we find S_∞?*
   (Use the formula $S_\infty = \frac{a}{1-r}$).
   Ask: Do you know the value of a and r?
   (Yes).
   Ask: How do we find S_n?
   (Use the formula $S_n = \frac{a(1-r^n)}{1-r}$)
   Point out again that learners already know the values of a and r.
   Once learners have substituted, tell them they need to simplify and according to the question, their answer should be in the form ab^n so they need to confirm this.

Solutions

1. $r = \frac{T_2}{T_1}$

$= \frac{5}{10} = \frac{1}{2}$

$T_5 = T_4 \times \frac{1}{2}$

$= 1{,}25 \times \frac{1}{2}$

$= 0{,}625$ (or $\frac{5}{8}$)

2. $T_n = 10\left(\frac{1}{2}\right)^{n-1}$

3. $r = \frac{1}{2}$ $\therefore$ the sequence converges because –1 < r < 1

4. $$S_\infty - S_n = \frac{a}{1-r} - \frac{a(1-r^n)}{1-r}$$

$$= \frac{10}{1-\frac{1}{2}} - \frac{10\left(1-\frac{1^n}{2}\right)}{1-\frac{1}{2}}$$

$$= 20 - 20\left(1-\frac{1^n}{2}\right)$$

$$= 20 - 20 + 20.\frac{1^n}{2}$$

$$= 20.\frac{1^n}{2}$$

Questions

Consider the series: $S_n = -3 + 5 + 13 + 21 + \ldots$ to n terms.

1. Determine the general term of the series in the form $T_k = bk + c$
2. Write S_n in sigma notation.
3. Show that $S_n = 4n^2 - 7n$
4. Another sequence is defined as:
   $Q_1 = -6$
   $Q_2 = -6 - 3$
   $Q_3 = -6 - 3 + 5$
   $Q_4 = -6 - 3 + 5 + 13$
   $Q_5 = -6 - 3 + 5 + 13 + 21$
   a) Write down a numerical expression for Q_6.
   b) Calculate the value of Q_{129}.

Teaching notes

1. Ask: *What kind of series is this?*
   (Arithmetic).
   Ask: *What is required to find the general term of an arithmetic series?*
   (The first term and common difference).
2. Ask: *What is required for sigma notation?*
   (The general term, the position of the first term being added and the position of the last term).
   Remind learners that the last term will be represented by a variable if the sigma notation is representing a general term as this one is.
3. Ask: *What is required to find S_n of an arithmetic sequence?*
   (The first term, the common difference and position),
   Point out that in this case, position will be a variable as it is a question regarding the general term.

4. a) Tell learners to note that they have not been asked to find the actual value of Q_6, only a numerical expression.
   Ask: *What do you notice about the series represented?*
   (It is the same series that was used in the previous three questions but -6 has been included now).
   b) As this is a series, learners need to realise that finding Q_{129} is a 'sum' question.
   Ask: *Could we use the formula for sum of an arithmetic series and find S_{129}?*
   (No, because the –6 is not part of a sequence with a common difference).
   Ask: *How can we get around this problem?*
   (Find S_{128} and then include the –6).
   Tell learners that to find S_{128} (Q_{128}), because the series that is now being used is the one that starts at –3 and is the same one from the beginning of the question, the answer for (3) can be used to find the sum.

Solutions

1. T_k = a + (k – 1)d
   T_k = –3 + (k – 1)8
   T_k = –3 + 8k – 8
   T_k = 8k – 11
2. $\sum_{k-1}^{n}(8k-11)$
3. $S_n = \frac{n}{2}[2a + (n-1)d]$
   $= \frac{n}{2}[2(-3) + (n-1)(8)]$
   $= \frac{n}{2}[-6 + 8n - 8]$
   $= \frac{n}{2}[8n - 14]$
   $= \frac{8n^2}{2} - \frac{14n}{2}$
   $= 4n^2 - 7n$
4. a) $Q_6 = -6 - 3 + 5 + 13 + 21 + 29$
   b) $Q_{129} = -6 + S_{128}$
   $= -6 + 4(128)^2 - 7(128)$
   $= 64\ 634$

1. Ask directed so that you can ascertain learners' level of understanding.
   Ask learners if they have any questions.

2. Give learners an exercise to complete on their own.

3. Walk around the classroom as learners do the exercise. Support learners where necessary.

TERM 1, TOPIC 1, LESSON 9

ASSIGNMENT

Suggested lesson duration: 1 hour

POLICY AND OUTCOMES

A

CAPS Page Number	40

Lesson Objectives

By the end of the lesson, learners will have:

- completed an assignment on sequences and series.

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the assignment. The assignment is Resource 1 in the Resource Pack.
3. Make copies of the assignment for each learner.

CONCEPTUAL DEVELOPMENT

C

INTRODUCTION

1. An assignment is generally an extended piece of work. Assignments can be used to consolidate or deepen understanding of work.
2. This assignment covers all the concepts from the topic, Sequences and Series.

DIRECT INSTRUCTION

1. Hand out the assignment to each learner.
2. Tell learners that they must work on their own and that they will have 1 hour to complete the assignment.
3. Tell learners the mark they receive will count 15% towards their school-based assessment mark for the end of the year.

MEMORANDUM

1. ASSIGNMENT **TOTAL: 60**

MEMORANDUM: SEQUENCES AND SERIES

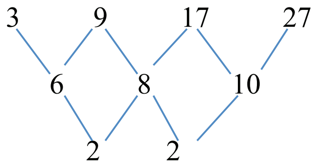
1.1	The sequence below can be used to determine the total number of coins in the 40th row: 1; 3; 5; 7… Arithmetic sequence $a = 1$ and $d = 2$ $T_{40} = ?$ $T_n = a + (n-1)d$ $T_{40} = 1 + (40-1)2$ $= 79$ OR $n = 40$, which is an even number ∴ Number of coins: $T_n = n - 1 + n$ $T_n = 2n - 1$ $T_{40} = 2(40) - 1$ $= 79$	✓ for $d = 2$ ✓substitution in correct formula ✓ answer ✓ $T_n = 2n - 1$ ✓substitution in correct formula ✓ answer (3)
1.2	$n = 40$, which is an even number $\therefore T_n = (n-1)(1) + n(5)$ Total value $= (40-1)(1) + (40)(5)$ $= 39 + 200$ $= 239$ **OR** Even rows form arithmetic sequence: 11; 23; 35; 42… $a = 11; d = 12;\ n = 20$ $T_n = a + (n-1)d$ $T_{20} = 11 + (20-1)(12)$ $= 239$	✓$(n-1)$ ✓$5n$ ✓substitution in correct formula ✓ answer ✓ for sequence ✓ $d = 12$ ✓substitution in correct formula ✓ answer (4)
1.3	If n is odd : $T_n = (n-1)(5) + n(1) = 6n - 5$ $T_n = 337$ $n = ?$ $(n-1)(5) + n(1) = 337$ $5n - 5 + n = 37$ $6n = 342$ $n = 57$ If n is even: $T_n = (n-1)(1) + n(5) = 337$ $6n - 1 = 337$ $6n = 338$ $n = 56.333\ldots$ or $56\frac{1}{3}$ **Not applicable** since $56\frac{1}{3}$ is not a natural number	$(n-1)(5) + n(1) = 337$ ✓ equation ✓simplifying ✓ answer $(n-1)(1) + n(5) = 337$ ✓ equation ✓ answer ✓ not applicable (6)

1.4	$S_{40} = 1 + 11 + 13 + 23 + 25 + 35 + \cdots + 239$	✓ for generating sequence ✓$(1 + 13 + 25 + \cdots)$
	$= (1 + 13 + 25 + \cdots) + (11 + 23 + 35 + \cdots)$ $= \frac{20}{2}[2 + (20 - 1)12] + \frac{20}{2}[22 + (20 - 1)2]$	✓ $11 + 23 + 35 + \cdots$
	$= 10(230) + 10(250)$ $= 4\,800$ cents	✓$\frac{20}{2}[2 + (20 - 1)12]$ ✓$\frac{20}{2}[22 + (20 - 1)2]$ ✓ answer
	OR Series for 1-cent coins: $S_{40} = 1 + 1 + 3 + 3 + 5 + 5 + \cdots + 39 + 39$	
	$= 2(1 + 3 + 5 + \cdots)$ $= 2S_{20}$ $a = 1, d = 2, n = 20$ $S_n = \frac{n}{2}[2a + (n - 1)d]$	✓ $2(1 + 3 + 5 + \cdots)$
	$S_{40} = 2\left[\frac{20}{2}(2 + (20 - 1)2)\right]$ $= 800$ coins	✓ 800 coins
	Series for 5-cent coins : $S_{40} = 0 + 2 + 2 + 4 + 4 + 6 + 6 + \cdots + 38 + 40$	✓ for generating sequence
	$= (0 + 2 + 4 + 6 + \cdots) + (2 + 4 + 6 + \cdots)$ Note: This is a combination of two arithmetic series: $0 + 2 + 4 + 6 + \cdots$; $a = 0$ and $d = 2$ $n = 20$ $2 + 4 + 6 + 8 + \cdots$; $a = 2$ and $d = 2$ $n = 2$	
	$\therefore S_{40} = \frac{20}{2}[0 + (20 - 1)2] + \frac{20}{2}[4 + (20 - 1)2]$	✓ for splitting sequence ✓ 800
	$= 380 + 420$ $= 800$ coins	✓4 800
	Total value is : $800(1) + 800(5) = 4\,800$ cents	(6)

2.		
2.1	$S_n = n(23 - 3n)$ $S_1 = 1[23 - 3(1)]$ $S_1 = 20$ $T_1 = 20$ $S_2 = 2[23 - 3(2)]$ $S_2 = 34$ $T_2 = S_2 - S_1$ $= 34 - 20$ $T_2 = 14$ $S_3 = 3[23 - 3(3)]$ $S_3 = 42$ $T_3 = 42 - 34$ $= 8$	✓substitution in formula $S_1 = 1[23 - 3(1)]$ ✓value $T_1 = 20$ ✓value $S_2 = 34$ ✓value $T_2 = 14$ ✓value $T_3 = 8$ (5)
2.2	$20; 14; 8; \ldots$ $a = 20; d = -6$ The sequence is arithmetic. $T_n = a + (n - 1)d$ $T_{15} = 20 + (14)(-6)$ $T_{15} = -64$	✓value of *d* ✓substitution in the correct formula ✓answer (3)
3.		
3.1	$ar^2 + ar = 280$…………….(1) $ar^4 + ar^5 = 4375$………….(2) $\frac{(1)}{(2)}$: $\frac{ar^4+ar^5}{ar+ar^2} = \frac{4375}{280}$ $\frac{(1)}{(2)}$: $\frac{ar^4(1+r)}{ar(1+r)} = \frac{4375}{280}$ $r^3 = \frac{125}{8}$ $r = \frac{5}{2}$	✓$ar^2 + ar = 280$ ✓ $ar^4 + ar^5 = 4375$ ✓ $\frac{ar^4 + ar^5}{ar + ar^2} = \frac{4375}{280}$ ✓ for common factor ✓$r^3 = \frac{125}{8}$ ✓ answer (6)

3.2	$ar^2 + ar = 280$ $a\left[\left(\frac{5}{2}\right)^2 + \frac{5}{2}\right] = 280$ $a\left(\frac{25}{4} + \frac{5}{2}\right) = 280$ $a\left(\frac{35}{4}\right) = 280$ $a = 32$	✓$a\left[\left(\frac{5}{2}\right)^2 + \frac{5}{2}\right] = 280$ Substitution of the value of *r* in equation (1) or (2) ✓ answer (2)
3.3	$S_n = \frac{a(r^n - 1)}{r - 1}$ $a = 32, \quad r = \frac{5}{2}$ or 2,5 $S_{10} = \frac{32\left[\left(\frac{5}{2}\right)^{10} - 1\right]}{2,5 - 1}$ $= 203429,19$	✓ substitution in the correct formula ✓ answer (2)
4.	$\sum_{r=1}^{\infty} 4.k^{r-1} = 5$ $4{+}4k + 4k^2 + 4k^3 \ldots = 5$ $S_\infty = \frac{a}{1 - r}$ $a = 4\,, \quad r = k, \quad S_\infty = 5$ $5 = \frac{4}{1 - k}$ $5 - 5k = 4$ $5k = 1$ $k = \frac{1}{5}$	✓for the series ✓ equating the series to 5 ✓$r = k$ ✓$5 = \frac{4}{1 - k}$ substitution in the formula ✓simplifying ✓ answer (6)

5.	$2(5)^5 + 2(5)^4 + 2(5)^3 + \cdots$ $r = \frac{T_2}{T_1}$ $r = \frac{2(5)^4}{2(5)^5}$ $r = \frac{1}{5}$ For convergence $-1 < r < 1$ Since : $-1 < \frac{1}{5} < 1$ This implies that the series converges.	$r = \frac{1}{5}$ ✓$-1 < \frac{1}{5} < 1$ (2)
6.	$2; x; 18$ $\frac{x}{2} = \frac{18}{x}$ $x^2 = 36$ $x^2 - 36 = 0$ $(x-6)(x+6) = 0$ $x = 6$ or $x = -6$ **OR** $\frac{x}{2} = \frac{18}{x}$ $x^2 = 36$ $x = \pm\sqrt{36}$ $x = \pm 6$	✓$\frac{x}{2} = \frac{18}{x}$ ✓$x^2 = 36$ ✓ for both factors ✓for both values of *x* (4) ✓$\frac{x}{2} = \frac{18}{x}$ ✓$x^2 = 36$ ✓$x = \pm\sqrt{36}$ $x = \pm 6$✓ 1 mark for both values of *x*

7.	$3^{n+1} > 20\ 000$ $n + 1 > log_3 20\ 000$ $n + 1 > \frac{\log 20\ 000}{\log 3}$ $n + 1 > 9{,}0145 ....$ $n > 8{,}0145 ...$ $n = 9$	✓$3^{n+1} > 20\ 000$ For the inequality ✓log form ✓$n + 1 > 9{,}0145$ Value of log $n > 8{,}0145$✓ Simplifying ✓answer
8.		
8.1	39	✓answer (1)
8.2	 $2a = 2$ $a = 1$ $c = 3 - 4 = -1$ $T_n = n^2 + bn - 1$ $3 = (1)^2 + b(1) - 1$ $b = 3$ $T_n = n^2 + 3n - 1$ **OR** $T_n = an^2 + bn + c$ $2a = 2$ $a = 1$ $3a + b = 6$ $3(1) + b = 6$ $b = 3$ $a + b + c = 3$ $1 + 3 + c = 3$ $c = -1$ $T_n = n^2 + 3n - 1$	✓$a = 1$ ✓$c = -1$ ✓formula ✓ $b = 3$ ✓formula ✓$a = 1$ ✓ $b = 3$ ✓$c = -1$
8.3	$n^2 + 3n - 1 > 269$ $n^2 + 3n - 270 > 0$ $(n + 18)(n - 15) > 0$ The first value of n is 16 The term is $16^2 + 3(16) - 1 = 303$	✓$n^2 + 3n - 1 > 269$ ✓factors ✓ $n = 16$ ✓ answer

Term 1, Topic 2: Topic Overview

FUNCTIONS

A

A. TOPIC OVERVIEW

- This is the second of five topics in Term 1.
- This topic runs for hour weeks (18 hours).
- It is presented over seven lessons.
- The approved textbooks approach this topic quite differently. For this reason, Functions and Functions: Exponential and Logarithmic have been combined into one overall topic. If you, as the teacher, wish to rearrange the lessons to suit your textbook, feel free to do so. Ensure that everything covered in the plans is still done.
- An approximate time has been allocated to each lesson (which will total 18 hours). For example, one lesson in this topic could take two school lessons. Plan according to your school's timetable.
- Functions and Graphs counts 23% of the final Paper 1 examination.
- Algebraic manipulation such as substitution and solving equations are important skills for this section.

Breakdown of topic into 7 lessons:

	Lesson title	**Suggested time (hours)**		**Lesson title**	**Suggested time (hours)**
1	Revision and definition of a function	2.5	5	Logarithms (rules and equations)	2.5
2	Investigation	1	6	Logarithmic functions	2
3	Inverse functions and their graphs	4	7	Revision and Consolidation	2
4	Revision of exponential laws and exponential functions	3			

SEQUENTIAL TABLE

B

GRADE 10 and Senior phase	GRADE 12
LOOKING BACK	**CURRENT**
● The concept of a function, where output depends on input ● Work with relationships between variables by using tables, graphs, words and formulae ● The four basic functions: linear, quadratic, hyperbola and exponential graphs ● Transformations on the above functions (the effect of a,p and q) An understanding of ● the exponential function ● the laws of exponents	● Introduce a more formal definition of a function ● Inverse functions ● Logarithmic functions ● Restricted domains to make an inverse function a function ● Problem solving and graph work for all the prescribed functions ● Understand the definition of a logarithm ● Logarithmic laws (in the context of real-life problems) ● Logarithmic functions

WHAT THE NSC DIAGNOSTIC REPORTS TELL US

C

According to **NSC Diagnostic Reports** there are a number of issues pertaining to Functions.

These include:

- Not understanding a restricted domain
- Not making the link between the domain of a function and the range of its inverse
- Confusing inverse of a function with derivative of a function.

It is important that you keep these issues in mind when teaching this section.

While teaching Functions, it is important to emphasise the properties of functions and their inverses. Learners need to be exposed to functions that have restricted domains.

D ASSESSMENT OF THE TOPIC

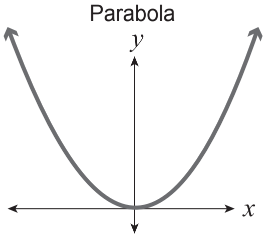
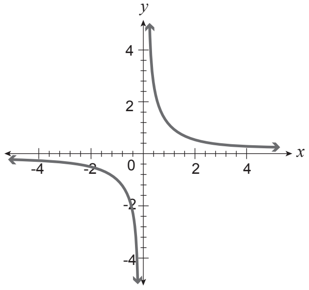
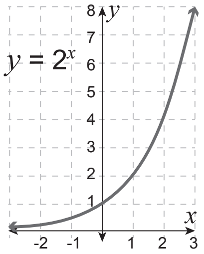
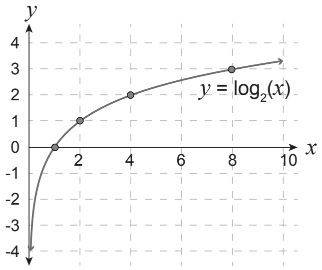
- CAPS formal assessment requirements for Term 1:
  - Investigation/Project
  - Assignment
  - Test
- Two tests, each with a memorandum, are provided in the Resource Pack (Resources 12-16). The tests are aligned to CAPS in every respect, including the four cognitive levels as required by CAPS (page 53).
- An investigation on Functions, with memorandum, is provided in the resource booklet.
- The questions usually take the form of sketches of functions which require interpretation which could include finding the equation of the function or could also require a learner to draw a function.
- Monitor each learner's progress to assess (informally) their grasp of the concepts. This information can form the basis of feedback to the learners and will provide you valuable information regarding support and interventions required.

E MATHEMATICAL VOCABULARY

Be sure to teach the following vocabulary at the appropriate place in the topic:

Term	**Explanation**
function	A relation for which each element of the domain (x) corresponds to exactly one element of the range (y). For every x-value there is only one possible y-value To test for a function on a graph, use the 'vertical line test' – run a ruler from left to right. If your ruler only ever touches the graph in one spot, it is a function
inverse function	A function that undoes the action of another function A function g is the inverse of a function f if whenever $y = f(x)$ then $x = g(y)$

TOPIC 2 FUNCTIONS

parabola	A function of the form $y = ax^2 + bx + c$ or $y = a(x-p)^2 +$ 
hyperbola	A function of the form $y = \frac{a}{x-p} + q$ 
exponential	A function of the form $y = ab^{x-p} + q$ 
logarithm	Mathematical operation that determines how many times a certain number, called the base, is multiplied by itself to reach another number For example, the base ten logarithm of 100 is 2, because ten raised to the power of two is equal to100
logarithmic function	A logarithmic function is the inverse of an exponential function 
increasing function	A function that is going ‘uphill’ when looking at it from left to right If a function has a turning point, this is the point at which a function could change from increasing to decreasing

decreasing function	A function that is going 'downhill' when looking at it from left to right If a function has a turning point, this is the point at which a function could change from decreasing to increasing
horizontal shift	A translation of the graph either to the left or the right
vertical shift	A translation of the graph either up or down
average gradient	The gradient between two points on a curved graph.
domain	All the possible x-values that are covered by the function/graph To find the domain always study the graph from left to right
range	All the possible y-values that are covered by the function/graph (To find the range always study the graph from bottom to top)
x-intercept	Where a function crosses (intercepts) the x-axis
y-intercept	Where a function crosses (intercepts) the y-axis
axis of symmetry	A line that cuts the function exactly in half A parabola has a vertical line of symmetry ($x = \ldots$) and a hyperbola has 2 axes of symmetry (which form a cross → X. These are in the form $y = mx + c$) An exponential graph does not have an axis of symmetry
asymptote	A straight line that a curved graph gets closer and closer to but never touches
turning point	The point at which a function (parabola, sin or cos graph) changes from increasing to decreasing OR from decreasing to increasing
maximum	The highest the graph can be and is always the y-value of the turning point Only involves functions that have a turning point
minimum	The lowest the graph can be and is always the y-value of the turning point Only involves functions that have a turning point

period	The distance required for the function to complete a full cycle This is given in degrees
convex	A convex curve is rounded like the exterior of a sphere or a circle (a convex curve is similar to a mountain) In a convex curve, a straight line joining any two points lies *totally* above the curve
concave	A concave curve rounds inward (a concave curve is similar to a valley) In a concave curve, a straight line connecting any two points on the curve lies entirely under the curve

TERM 1, TOPIC 2, LESSON 1

REVISION OF THE CONCEPT OF A FUNCTION

Suggested lesson duration: 2,5 hours

A POLICY AND OUTCOMES

CAPS Page Number	40, 41
Lesson Objectives By the end of the lesson, learners will have revised: ● functions from previous years and should be able to: ● define and recognise when a relation is a function or not.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Have Resource 2 ready for use during this lesson.
4. Write the lesson heading on the board before learners arrive.
5. Write work on the chalkboard before the learners arrive. For this lesson have the four sketches from point 1 ready.
6. If there isn't a revision exercise in the textbook that you use, either use the revision exercise at the end of a Grade 11 textbook or items from a Grade 11 test on Functions.
7. The tables below provide references to this topic in Grade 12 textbooks. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

Revision of Functions

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		CLASSROOM MATHS		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
				Qu's	38	2.1	45	2.2	48	2.1	52

TOPIC 2, LESSON 1: REVISION OF THE CONCEPT OF A FUNCTION

Definition of a Function

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		CLASSROOM MATHS		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
1	40	1	33	1 2	42 43			2.1	43	2.2	55

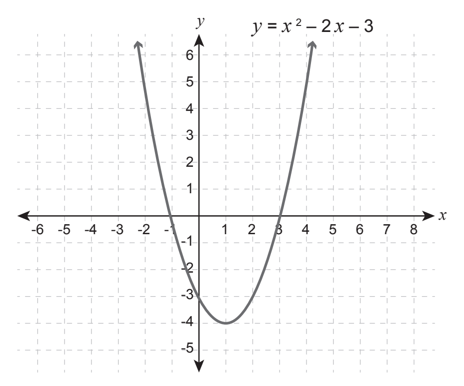
CONCEPTUAL DEVELOPMENT

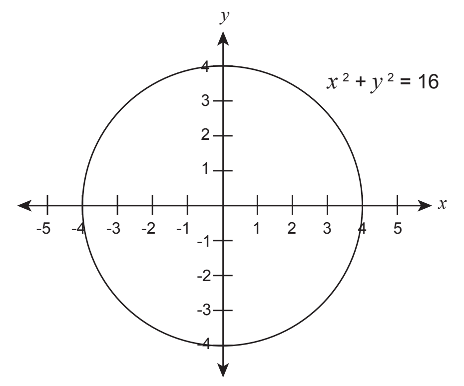
INTRODUCTION

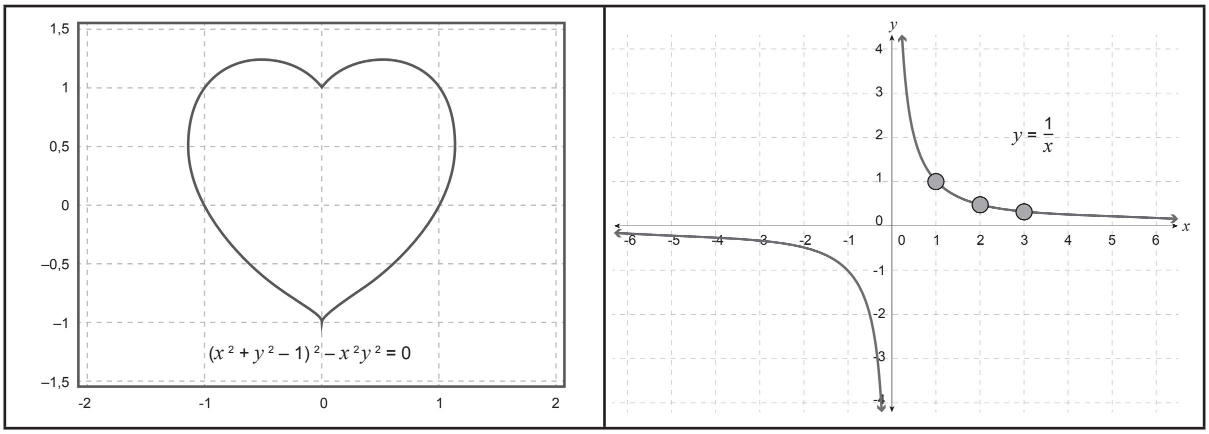
1. This lesson starts with revision of the formal definition of a function.
2. Discuss and use the vocabulary that is important in this section. Learners must be familiar with the correct terms. Continue to use the vocabulary revised in this lesson throughout this section. Question learners on a regular basis by asking them what each term means, particularly when doing examples that can be related to.
3. Learners should be confident in all aspects of the four functions, linear, the parabola (quadratic function), the hyperbola and the exponential function.
4. All aspects that have been covered before need to be revised and reinforced at this stage.
5. It is <u>not</u> a good idea for learners to draw functions using the table function on their calculators. A deeper understanding of the function is essential first.

DIRECT INSTRUCTION

1. Start the lesson by drawing the following relations and functions on the board:







2. Tell learners that two of these sketches represent functions and two do not.
   Ask: *Which two are the functions?*
   (the parabola and the hyperbola)

3. Ask: *Why aren't the other two sketches (circle and a heart) functions?*
   (There are 2 y-values for some of the x-values. In a function there is only one possible y-value for each x-value).

4. Demonstrate this with a ruler held parallel to the y-axis (remind learners that this is known as the vertical line test). While discussing each relation's graph, stop once or twice at a specific x-value and ask what the corresponding y-value(s) is/are.

5. First consider the parabola: Show that, as the ruler moves slowly from left to right it only ever crosses one point on the function. This shows that there is only one possible y-value for an x-value.

6. Repeat with the circle (tell learners that they will be learning about this in Analytical Geometry later in the year): Show that, as the ruler moves slowly from left to right it often crosses two points on the graph. This shows that there is more than one possible y-value for an x-value.

7. Repeat with the heart: Show that, as the ruler moves slowly from left to right it often crosses two points on the graph. This shows that there is more than one possible y-value for an x-value.

8. Repeat with the hyperbola: Show that, as the ruler moves slowly from left to right it only ever crosses one point on the function. This shows that there is only one possible y-value for an x-value.

9. Give learners a definition of a function to write in their books:
   A function is a relation for which each element of the domain (x) corresponds to exactly one element of the range (y). For every x-value there is only one possible y-value.

10. Write the following functions on the board:

$f(x) = x^2 + 2x - 3$	$g(x) = \frac{-3}{x-2}$	$h(x) = 2^x - 1$

11. Say: *Understanding function notation is important. Find $f(3)$, g(–1) and $h(4)$*

$f(x) = x^2 + 2x - 3$	$g(x) = \frac{-3}{x-2}$	$h(x) = 2^x - 1$
$f(3) = 12$	$g(-1) = 1$	$h(4) = 15$

12. Say: *Explain what was asked and what was found in these three questions.*
    (The value y was found after being given the value of x. The point found is a point on the graph).

13. Say: *In other words, (3;12) is a coordinate on the parabola represented by $f(x)$ and (–1;1) is a coordinate on the hyperbola represented by g(x) and (4;15) is a coordinate on the exponential graph represented by $h(x)$.*

14. Say: *Similarly, we can find the value of x, given the y-value.*

15. Write the following on the board:

<table>
<tr><td>If $f(x) = 5$, find the value(s) of x</td><td>If $g(x) = -3$, find the value(s) of x</td><td>If $h(x) = 1$, find the value(s) of x</td></tr>
<tr><td colspan="3">Tell learners to note that the entire function has now been given a value. In other words, the y-value has been given. To find the possible x-values, solving an equation will be required. Ask learners to find their answers on their own before you do the work on the board.</td></tr>
<tr><td>$f(x) = x^2 + 2x - 3$
$f(x) = 5$
$5 = x^2 + 2x - 3$
$0 = x^2 + 2x - 8$
$0 = (x + 4)(x - 2)$
$x = -4$ or $x = 2$</td><td>$g(x) = \frac{-3}{x-2}$
$g(x) = -3$
$-3 = \frac{-3}{x-2}$
$-3(x - 2) = -3$
$-3x + 6 = -3$
$-3x = -9$
$x = 3$</td><td>$h(x) = 2^x - 1$
$h(x) = 1$
$1 = 2^x - 1$
$2 = 2^x$
$\therefore\ x = 1$</td></tr>
<tr><td colspan="3">Refer learners back to the sketches to confirm that all the possible points found make sense.</td></tr>
</table>

16. Ask if anyone has any questions on functions and the notation used before moving on.

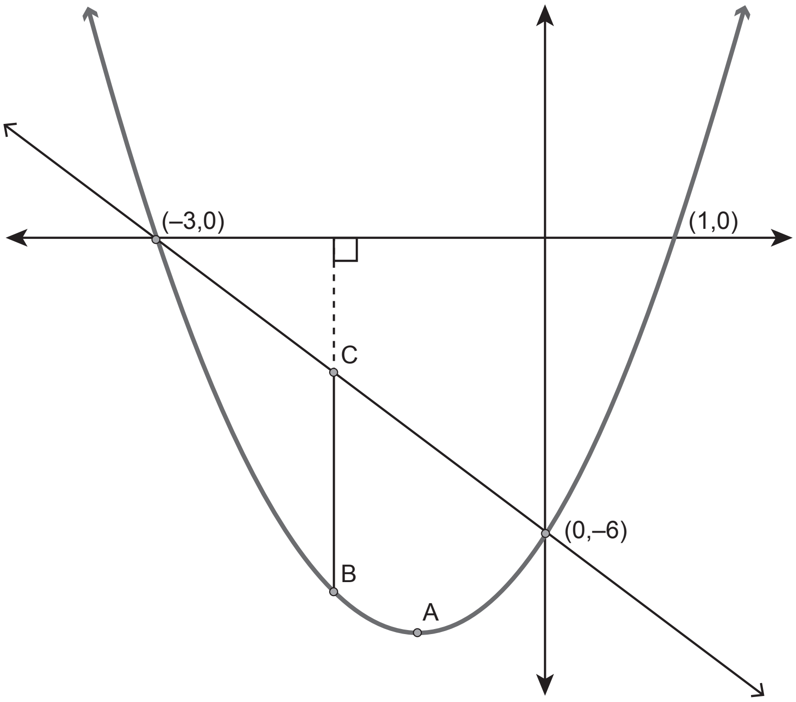
17. The rest of this lesson is made up of three fully worked examples from Grade 11 assessments. As you work through these with the learners, it is important to frequently talk about as many concepts as possible.

For example, use the words increasing, decreasing, positive, negative, domain, range (and so on) wherever possible even if they are not always part of the question.

18. Ask learners to regularly stop and check if their answers 'look' reasonable. Learners should always look back at the sketch and ask themselves whether they expect a positive/negative answer; or does it look like the graph is increasing at the correct values of x and so on.
19. Do several fully worked examples with learners now. Learners should write the examples in their books, making notes as they do so.
    Do not tell learners everything in the 'Thought process/discussion' column. Use the points to ask them questions to encourage thinking on their part.

Examples

1. Given $f(x) = 2x^2 + 4x - 6$ and $g(x) = -2x - 6$



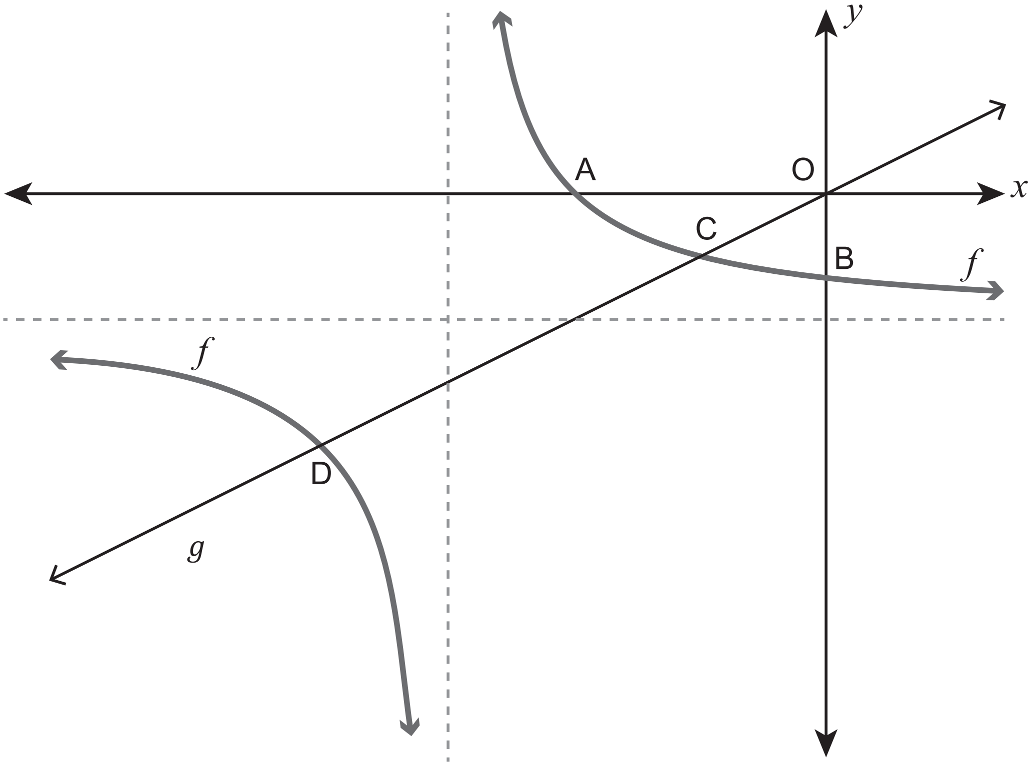
Question	Teaching notes
a) Determine the average gradient from the turning point to the y-intercept of that parabola	To find the gradient, we need two points. One is given. The y-intercept is (0;–6) To find the other point required: $x = \frac{-b}{2a}$ Substitute this value into the equation to find the corresponding y-value Use $m = \frac{y_2 - y_1}{x_2 - x_1}$ to find the average gradient.
b) Find an expression to represent the length of CB	Expression of (top graph minus bottom graph) and simplify.

c) Find the maximum possible length of CB.	Find the y-co-ordinate of the turning point of the expression found in b).
d) For what value of x is $f(x) < g(x)$?	First read the question as: Parabola less than straight line This means: where is the parabola underneath the straight line Recommend colouring or highlighting this section. Now focus on the corresponding x-values You may need to remind learners when to use round (excluding) or square brackets (including).
e) Find the equation of $h(x)$, the reflection of $f(x)$ in the y-axis.	To reflect in the y-axis, the x-value changes signs and the y–value remains the same. Perform this rule on the function's equation and simplify.
Solutions	
a) $x = \frac{-b}{2a}$ $x = \frac{-(4)}{2(2)}$ $x = -1$ $\therefore y = 2(-1)^2 + 4(-1) - 6$ $\therefore y = -8$ Turning point: (-1 ;-8) y-intercept: (0 ;-6) Average gradient: $m = \frac{-6-(-8)}{0-(-1)}$ $m = 2$	b) CB = $-2x - 6 - (2x^2 + 4x - 6)$ CB = $-2x - 6 - 2x^2 - 4x + 6$ CB = $-2x^2 - 6x$
c) $CB = -2x^2 - 6x$ $x = \frac{-b}{2a}$ $x = \frac{-(-6)}{2(-2)}$ $x = \frac{-3}{2}$ $\therefore y = -2\left(\frac{-3}{2}\right) - 6\left(\frac{-3}{2}\right)$ $\therefore y = \frac{9}{2}$ The maximum length of CB is $\frac{9}{2}$ units	d) $x \in (-3 ; 0)$ e) $h(x) = 2(-x)^2 + 4(-x) - 6$ $h(x) = 2(-x)^2 - 4x - 6$

2. The diagram below shows the graph of $f(x) = \frac{1}{x+3} - 1$ and $g(x) = \frac{1}{2}x$

The graph of f intersects the x-axis at A and the y-axis at B.

The graph of f and g intersect at points C and D.

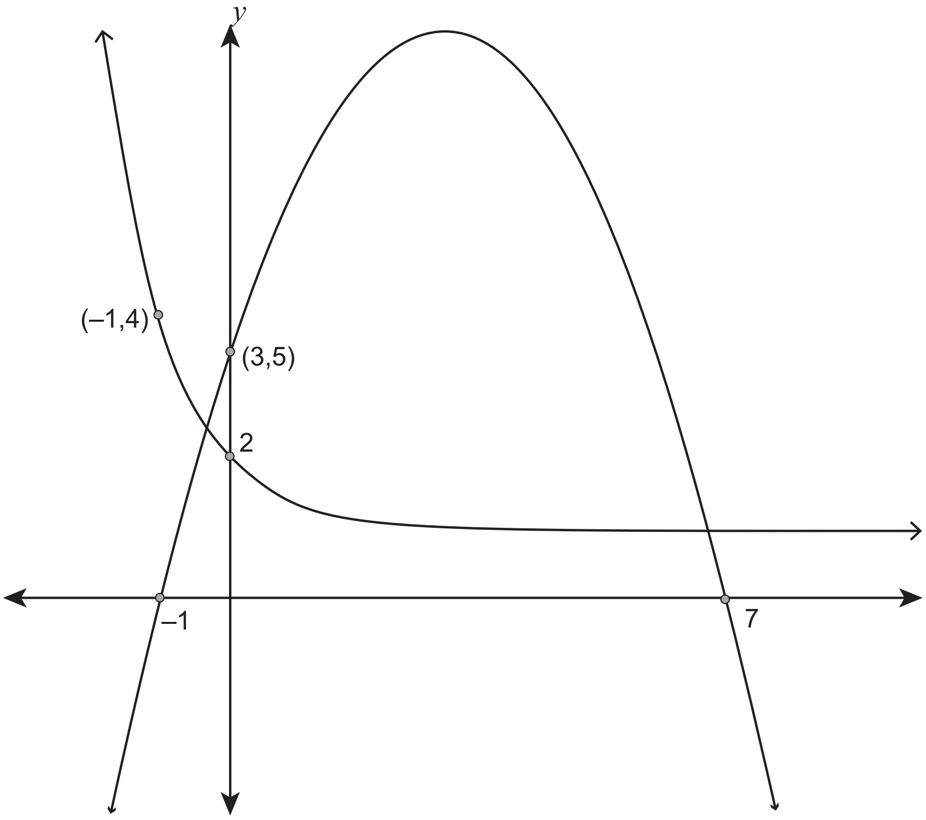


Question	Teaching notes
a) Write down the equations of the asymptotes of f	Point out the different format that this hyperbola takes – it has a '+' in the denominator. Remind learners that the asymptote is the value of x that makes the denominator equal to zero.
b) Determine the domain of f	Ask: *What is the domain and range?* (Domain — all the possible x-values Range — all the possible y-values). Ask a learner to come to the board and show how he/she would find the domain of the hyperbola (using a ruler and move from left to right with a focus on the x-values). Discuss how it was not possible to keep the ruler running along but the need to 'hop' over the asymptote was necessary and therefore showed that there was a value that was not possible for x.

c) Calculate the length of OB and OA	Ask: *What is 'happening' at B and A?* (B is the y-intercept and A is the y-intercept). Therefore, these values will need to be found in order to find the lengths. Remind learners that lengths are always positive.
d) Determine the co-ordinates of C and D	Ask: *What is 'happening' at C and D?* (They are the two points of intersection of the two graphs). To find the point of intersection, make the two graphs equal to each other and solve. Once the values of x have been found, substitute them into any of the other two equations and find the corresponding y-values.
e) Use the graphs to obtain the solution to $\frac{1}{x+3} \geq \frac{x+2}{2}$	This is a difficult question that many learners will struggle with. Ask learners if they can see any connection between the two expressions in the question and the functions given in the question. The one on the left is the hyperbola without the '–1'. Ask if they can 'find' the '–1' in the expression on the right-hand side (but point out that on the RHS it would be a '+1'). Expand $\frac{x+2}{2}$ into $\frac{x}{2} + \frac{2}{2} = \frac{x}{2} + 1$ Now re-write the question to make it easier to deal with: $\frac{1}{x+3} \geq \frac{x+2}{2}$ $\frac{1}{x+3} \geq \frac{x}{2} + 1$ $\frac{1}{x+3} - 1 \geq \frac{x}{2}$ The question is actually asking: for what values of x Is the hyperbola above or equal to the straight line?

Solutions	
a) $x = -3$ $y = -1$	b) $x \in R$; $x \neq -3$
c) $f(x) = \frac{1}{x+3} - 1$ x-int ; $y = 0$ $0 = \frac{1}{x+3} - 1$ $1 = \frac{1}{x+3}$ $x + 3 = 1$ $x = -2$ (–2;0) $\therefore$ $OA = 2$ units y-int ; $x = 0$ $y = \frac{1}{0+3} - 1$ $y = \frac{1}{3} - 1$ $y = \frac{-2}{3}$ $(0;\frac{-2}{3})$ $\therefore$ $OB = \frac{2}{3}$ units	d) $\frac{1}{x+3} - 1 = \frac{1}{2}x$ LCD $2(x + 3)$ $2 - 2(x + 3) = x(x + 3)$ $2 - 2x - 6 = x^2 + 3x$ $0 = x^2 + 5x + 4$ $0 = (x + 1)(x + 4)$ $\therefore$ $x = -1$ and $x = -4$ Substitute into $y = \frac{1}{2}x$ $y = \frac{1}{2}(-1)$ $y = -\frac{1}{2}$ $y = \frac{1}{2}(-4)$ $y = -2$ The two points are: $(-1 ; \frac{-1}{2})$ $(-4 ; -2)$ $\therefore$ $C(-1 ; \frac{-1}{2})$ $D(-4 ; -2)$ Ensure that learners look carefully at the sketch to check which points match C and D.
e) $\frac{1}{x+3} \geq \frac{x+2}{2}$ $\frac{1}{x+3} \geq \frac{x}{2} + 1$ $\frac{1}{x+3} - 1 \geq \frac{x}{2}$ $f(x) \geq g(x)$ $\therefore$ $x \leq -4$ or $-3 < x \leq -1$	Note regarding (e): Point out to learners the use of the < sign at the -3 instead of the ≤ sign. Ensure they understand it is because of the asymptote.

3. The graphs of the form $f(x) = ax^2 + bx + c$ and $g(x) = d^x + e$ are sketched below. The y-intercept of the parabola is (0 ; 3.5) and the x-intercepts are (–1 ; 0) and (7 ; 0)



Question	Teaching notes
a) Find the values of a, b and c	As the x-intercepts and one other point is given, use $y = a(x - x_1)(x - x_2)$ to find the equation of the parabola and hence find a, b and c. Remind learners that they should check if their answers make sense. a) Easy to check – should it be positive or negative? c) Easy check - it should be the y-intercept.
b) Find the values of d and e	Substitute the y-intercept to find 'e' ($d^o = 1$ so the 'd' will no longer be in the equation so 'e' will be found) Use the other point and the value of 'e' to find 'd'.
c) Determine the co-ordinate of the turning point of $f(x)$	$x = \frac{-b}{2a}$ Substitute this value into the equation to find the corresponding y-value.

d) Find the range of $f(x)$	Ask the class what domain (all the possible x-values) and range (all the possible y-values) mean. Ask a learner to come to the board and show how he/she would find the range of the parabola (using a ruler and move from bottom to top with a focus on the y-values). You may need to remind learners when to use round (excluding) or square brackets (including).
e) Determine the equation of the axis of symmetry of $f(x)$	Use $x = \frac{-b}{2a}$ or symmetry from the x-intercepts.
f) Determine the equation of the asymptote of $g(x)$	This has essentially already been found in b). Remind learners to write the equation properly ($y = \dots$) and not just put the number down.
g) Determine the values of x for which the graph $f(x)$ increases	Ask a learner to come to the board and show where the parabola is increasing. Remind learners that it would be a good idea to highlight or colour this part of the graph a different colour. Ask the class to tell you what x-values match with this part of the graph. Remind learners that the graph cannot be increasing or decreasing at the turning point and this is why a round bracket is used to show where the list ends.
h) Determine the equation of $h(x)$, which is the graph formed by moving $g(x)$ 2 units to the left and 5 units down.	In order to answer this question, learners need to have a good understanding of which part of a function's equation represents the horizontal shift and the vertical shift. Ask learners to tell you where these transformations are shown in an equation.

Solutions	
a) $y = a(x - x_1)(x - x_2)$ $y = a(x - (-1))(x - 7)$ $y = a(x + 1)(x - 7)$ Substitute (0 ; 3.5) $3.5 = a(0 + 1)(0 - 7)$ $3.5 = a(1)(-7)$ $3.5 = -7a$ $\frac{-1}{2} = a$ $y = a(x + 1)(x - 7)$ $y = \frac{-1}{2}(x + 1)(x - 7)$ $y = \frac{-1}{2}(x^2 - 6x - 7)$ $y = \frac{-1}{2}x^2 + 3x + \frac{7}{2}$ $\therefore\ a = \frac{-1}{2} \quad b = 3 \quad c = \frac{7}{2}$	b) $y = d^x + e$ Substitute (0 ; 2) $2 = d^0 + e$ $2 = 1 + e$ $1 = e$ Substitute (–1 ; 4) $4 = d^{-1} + 1$ $3 = d^{-1}$ $3 = \frac{1}{d}$ $3d = 1$ $d = \frac{1}{3}$
c) $x = \frac{-b}{2a}$ $x = \frac{-3}{2\left(\frac{-1}{2}\right)}$ $x = 3$ $y = \frac{-1}{2}(3)^2 + 3(3) + \frac{7}{2}$ $y = 8$ Turning point: (3 ;8)	d) $y \in (-\infty ; 8]$
e) $x = 3$	f) $y = 1$
g) $x \in (-\infty ;3)$	h) $h(x) = \left(\frac{1}{3}\right)^{x+2} + 1 - 5$ $h(x) = \left(\frac{1}{3}\right)^{x+2} - 4$

20. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

21. Give learners an exercise to complete on their own. At this stage learners should feel confident in the four basic functions and be able to draw them or answer questions on functions that have already been sketched. If this is not the case, consider spending more time on Grade 10 and 11 concepts – if time in class is too tight then schedule an extra lesson.

22. Walk around the classroom as learners do the exercise. Support learners where necessary.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=OxZ0JL4Bjzk

https://www.youtube.com/watch?v=AvUlyCzhYBs

TERM 1, TOPIC 2, LESSON 2

INVESTIGATION

Suggested lesson duration: 1 hour

POLICY AND OUTCOMES

A

CAPS Page Number	40, 41
Lesson Objectives By the end of the lesson, learners will have: ● completed an investigation on functions and their inverses.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the investigation.
3. Make copies of the investigation for each learner. The investigation is Resource 3 in the Resource Pack.

CONCEPTUAL DEVELOPMENT

C

INTRODUCTION

1. An investigation is an activity that should lead a learner to a deeper understanding of a mathematical concept.
2. This investigation deals with functions and their inverses.

DIRECT INSTRUCTION

1. Hand out the investigation to each learner.
2. Tell learners that they must work on their own and that they will have 1 hour to complete the investigation.
3. Tell learners the mark they receive will count 15% towards their school-based assessment mark for the end of the year.

MARKING MEMORANDUM

MEMORANDUM: FUNCTIONS AND INVERSES

PART 1		
1.1		
1.1.1	One-to-many relation	✓answer (1)
1.1.2	One-to-one relation	✓answer (1)
1.1.3	Many-to-one relation	✓answer (1)
1.2		
a)	Not a function, for one input-value there are more than one output values.	✓answer ✓reason (2)
b)	Function, for one input value there is only one output-value.	✓answer and reason (1)
c)	Function, for more than one input value there is one output-value.	✓answer and reason (1)
1.3	*a*: Not a function *b*: Function *c*: Not a function *d*: Not a function *e*: Function *f*: Function *g*: Function *h*: Not a function	✓one mark for each answer (8)
PART 2		
2.1	$\begin{array}{|c|c|c|c|c|c|c|} \hline \boldsymbol{x} & -3 & -2 & -1 & 0 & 1 & 2 \\ \hline \boldsymbol{y} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 & 2 & 4 \\ \hline \end{array}$	✓ one mark for all *y*-values (1)

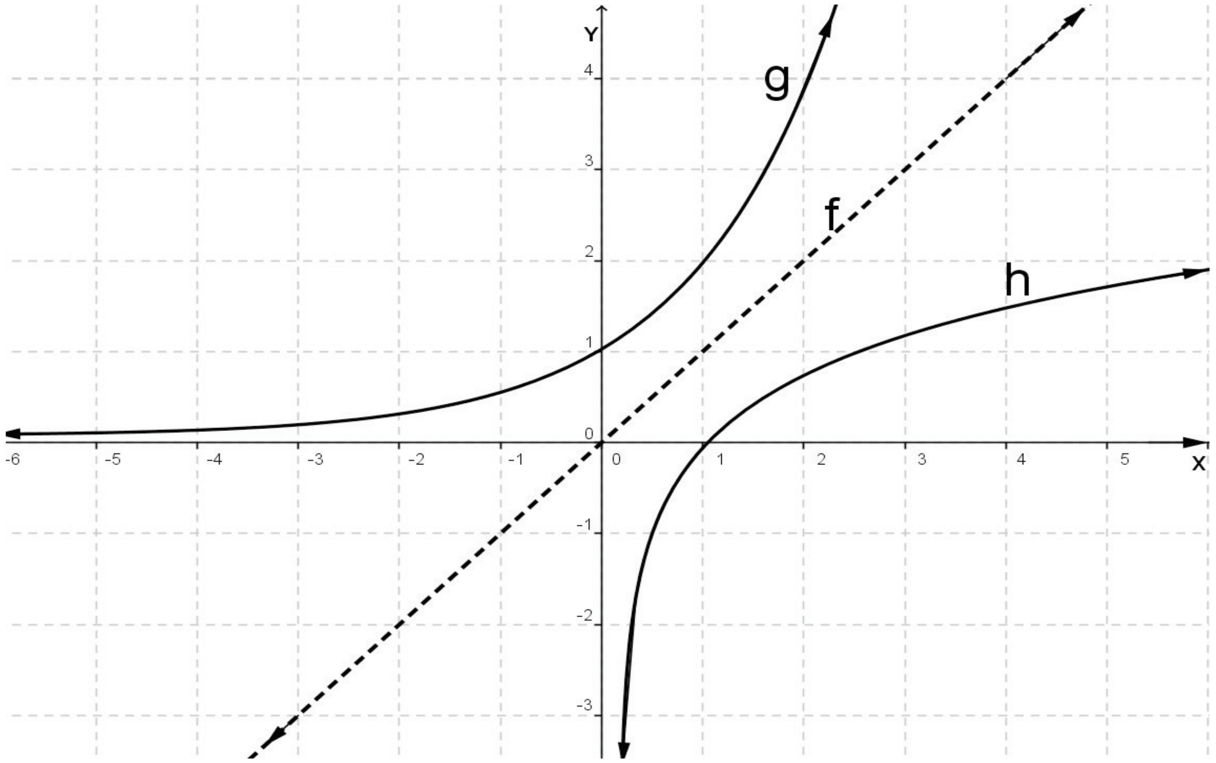
2.2

g: ✓ y-intercept
✓ shape
✓ asymptote

2.3

f: ✓ for both x- and y-intercepts

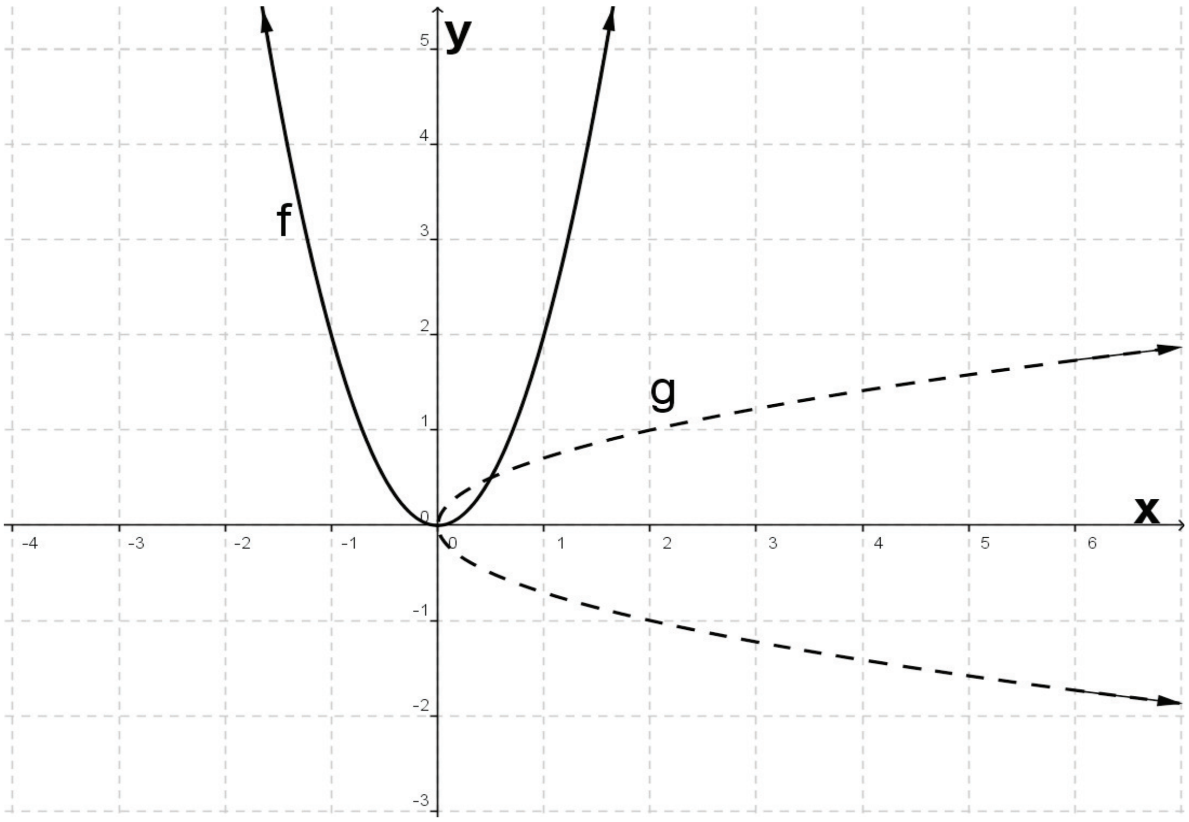
h: ✓ for x-intercept
✓ for asymptote



(6)

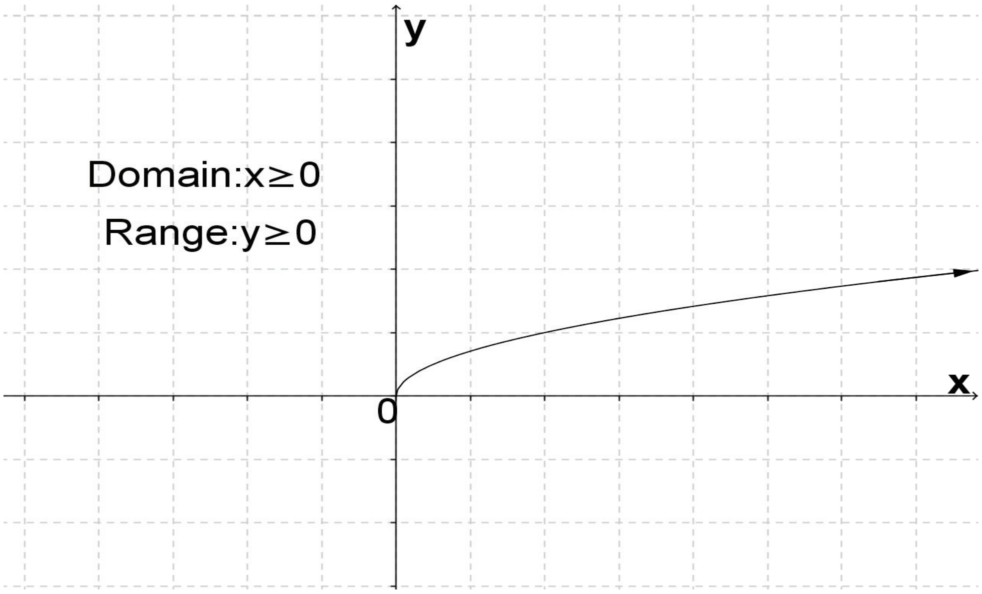
<table>
<tr><td>2.4</td><td><table><tr><td><i>y</i></td><td>−3</td><td>−2</td><td>−1</td><td>0</td><td>1</td><td>2</td></tr><tr><td><i>x</i></td><td>$\frac{1}{8}$</td><td>$\frac{1}{4}$</td><td>$\frac{1}{2}$</td><td>1</td><td>2</td><td>4</td></tr></table>
See sketch above for <i>h</i>.</td><td>✓ one mark for all <i>y</i>-values
✓ one mark for all <i>x</i>-values
(2)</td></tr>
<tr><td>2.5</td><td></td><td></td></tr>
<tr><td>2.5.1</td><td>No <i>x</i>-intercept</td><td>✓ answer
(1)</td></tr>
<tr><td>2.5.2</td><td>$x = 1$</td><td>✓ answer
(1)</td></tr>
</table>

2.6		
2.6.1	Domain: $x \in \mathbb{R}$ **OR** $-\infty < x < \infty$ **OR** $x \in (-\infty; \infty)$ Range: $y > 0$ **OR** $0 < y < \infty$ **OR** $y \in (0; \infty)$	✓ answer ✓ answer (2)
2.6.2	Domain: $x > 0$ **OR** $-\infty < x < \infty$ **OR** $x \in (-\infty; \infty)$ Range: $y \in \mathbb{R}$ **OR** $-\infty < y < \infty$ **OR** $y \in (-\infty; \infty)$	✓ answer ✓ answer (2)
2.6.3	The domain of one graph is the range of the other, and vice versa.	✓ answer (1)
2.6.4	Yes, both graphs are functions. They both pass the vertical-line test.	✓ answer ✓ reason (2)

2.6.5	$y = \log_2 x$	✓ answer (1)
2.6.6	Yes, the line with equation $y = x$ is the line of symmetry.	✓ answer ✓ reason (2)
2.6.7	Any relevant logical conjecture	(3)
PART 3		
3.1.1	$x = 2y^2$ or $y = \pm\sqrt{\frac{x}{2}}$	✓answer (1)
3.1.2	Turning point of f is (0; 0) and the turning point of the inverse is (0; 0)	✓ answer ✓ answer (2)
3.1.3		✓ for f ✓ the inverse g (2)
3.1.4	The inverse of f is not a function; it fails the vertical-line test.	✓answer ✓ reason (2)
3.1.5	$f(x) = 2x^2$, domain: $x \geq 0$ OR $x \in [0; \infty)$ $f(x) = 2x^2$, domain: $x \leq 0$ OR $x \in (-\infty; 0]$	✓one mark for each domain (2)
3.1.6	a) If the domain of f is $x \leq 0$, then the range of the inverse will be $y \leq 0$ b) If the domain of f is $x \geq 0$, then the range of the inverse will be $y \geq 0$	✓ for $x \leq 0$ and $y \leq 0$ ✓ for $x \geq 0$ and $y \geq 0$ (2)

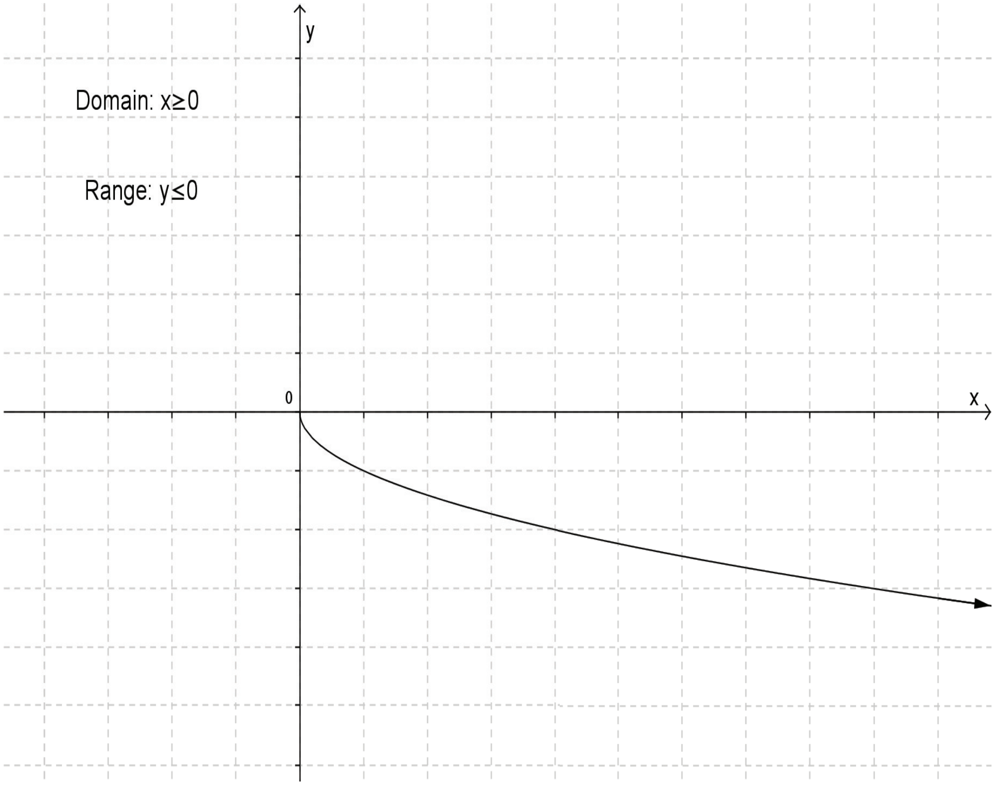
3.1.7

✓ domain
✓ correct shape



(2)

✓ domain
✓ correct shape



(2)

NB: The notation $f^{-1}(x) = \cdots$ is used only for one-to-one relations and may not be used for inverses of many-to-one relations as their inverses are not functions.

TERM 1, TOPIC 2, LESSON 3

INVERSE FUNCTIONS AND THEIR GRAPHS

Suggested lesson duration: 4 hours

POLICY AND OUTCOMES

A

CAPS Page Number	40, 41

Lesson Objectives

By the end of the lesson, learners should be able to:

- explain what an inverse function is
- find the equation of an inverse function
- restrict the domain of a quadratic function to make the inverse a function.

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Have Resource 4 ready for use during lesson.
4. Write the lesson heading on the board before learners arrive.
5. Write work on the chalkboard before the learners arrive. For this lesson have three Cartesian planes drawn, each with the line $y = x$ drawn in (points 2 and 4).
6. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**CLASSROOM MATHS**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
2 3	43 46	2 3 Rev	37 40 41	3 – 7 Qu's	44- 53 54	2.3 2.4 Rev	51 56 58	2.2 2.3 2.4 2.5	47 50 52 54	2.3 2.4 2.5	61 66 69

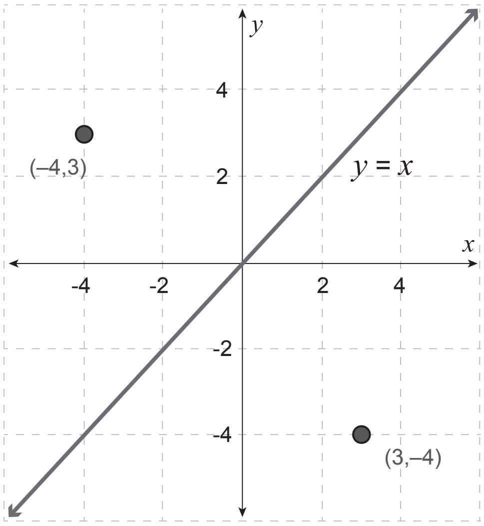
C CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. The textbooks approach this topic quite differently. It is important that you look carefully at your textbook and plan when you should stop teaching and give learners an exercise to do. The number of exercises available ranges from two to six.
2. There is no statement at the end of the lesson plan to give learners an exercise to do due to this – ensure your learners have completed all the exercises in the textbook used before moving on to lesson 4 at the appropriate place where a concept has been taught and needs to be practiced.
3. An entire week is spent on this lesson. Use the time well.
4. Learners have just completed an investigation on functions and their inverses – this should mean that when they are taught new concepts in this lesson they should already be a little familiar with it. That being said, teach it as if they have not completed the investigation – as thoroughly as possible.

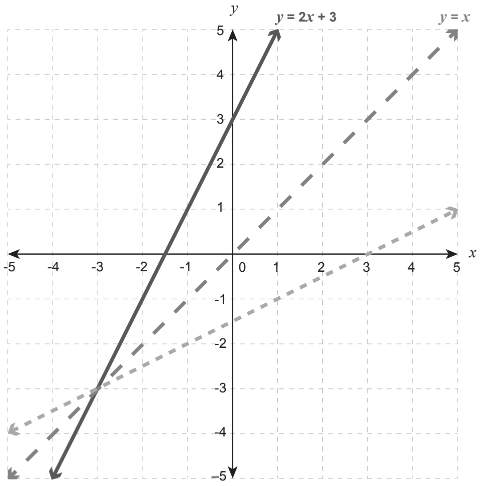
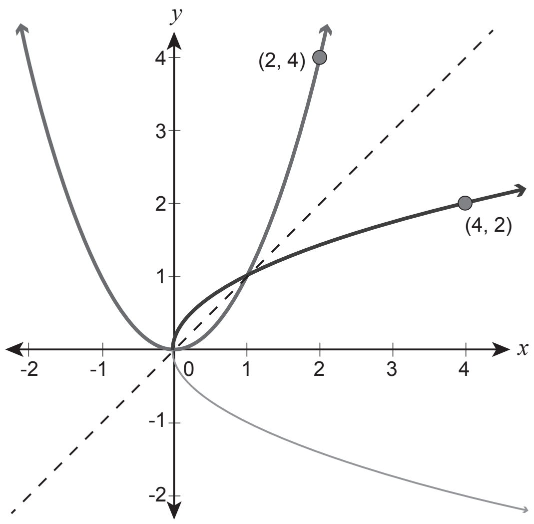
DIRECT INSTRUCTION

1. Start the lesson by telling learners that although they have just completed an investigation on functions and their inverses, you are going to teach inverse functions in detail now.
2. Draw a Cartesian plane on the board with the line $y = x$ drawn in. Ask learners to copy it into their books. Plot the point (-4;3). Tell learners to plot the point and its reflection in the line $y = x$.



3. Confirm that learners have plotted it in the correct place. Say: When a point is reflected in the line $y = x$, note that the x and y coordinates are interchanged.

4. Show learners the following two sketches:

	Ask learners to note the following for the linear function: ● One function is the reflection of the other function in the line $y = x$ (In other words, if a fold was made along the line $y = x$, the graphs would land on top of one another) ● The coordinates have changed according to the rule $(x\ ;\ y) \rightarrow (y\ ;\ x)$ ● This includes the x-intercept and y-intercept (point these out)
	Learners should note the following for the quadratic function: ● One graph is the reflection of the other function in the line $y = x$ ● The coordinates have changed according to the rule $(x\ ;\ y) \rightarrow (y\ ;\ x)$ ● Although this includes the x-intercept and y-intercept it isn't so easy to see as they were both zero originally. Ask: *What is the domain and range of the quadratic function passing through the point* (2 ; 4)*?* Confirm that learners know and understand that: $x \in \mathrm{R}$ and $y > 0$ Ask: *What is the domain and range of the quadratic relation passing through the point* (4 ; 2)*?* $y \in \mathrm{R}$ and $x > 0$

5. The question regarding the domain and range is an important one. Ensure that learners notice that they shouldn't have had to look at the sketch to answer the question for the domain and range of the reflected graph. It should have been clear that it was the opposite to the domain and range of the original function because when a function is reflected all x-values become the y-values and all y-values become the x-values.

6. Tell learners that what they have encountered in these two examples are inverse functions/relations.

7. Give learners the following definition to write in their books:

An inverse function is a function that undoes the action of another function. A function g is the inverse of a function f if, whenever $y = f(x)$ then $x = g(y)$

8. Use the above two examples to explain further. Start by discussing the linear function and its inverse.

9. Tell learners that the original linear function's equation is: $y = 2x + 3$. To find the inverse function:

Interchange the x-values and y-values	$x = 2y + 3$
Say: *Although this is the equation for the inverse function, it is not in standard form. A linear function should be in the form:* $y = mx + c$	$2y = x - 3$ $\frac{2y}{2} = \frac{x}{2} - \frac{3}{2}$ $y = \frac{1}{2}x - \frac{3}{2}$
Note the correct notation for inverse functions	If $f(x) = 2x + 3$ then $f^{-1}(x) = \frac{1}{2}x - \frac{3}{2}$

10. Refer learners back to the sketch of the two functions and note that:
    - The y-intercept of the original function is 3 and the x-intercept is $-\frac{3}{2}$
    - The y-intercept of the inverse function is $-\frac{3}{2}$ and the x-intercept is 3

11. Say: *Note that I called the inverse graph of the function an inverse function.*
    Ask: *Is this true? Is it a function?*
    (Yes – every x-value has only one possible y-value).

12. Discuss the quadratic function and its inverse.

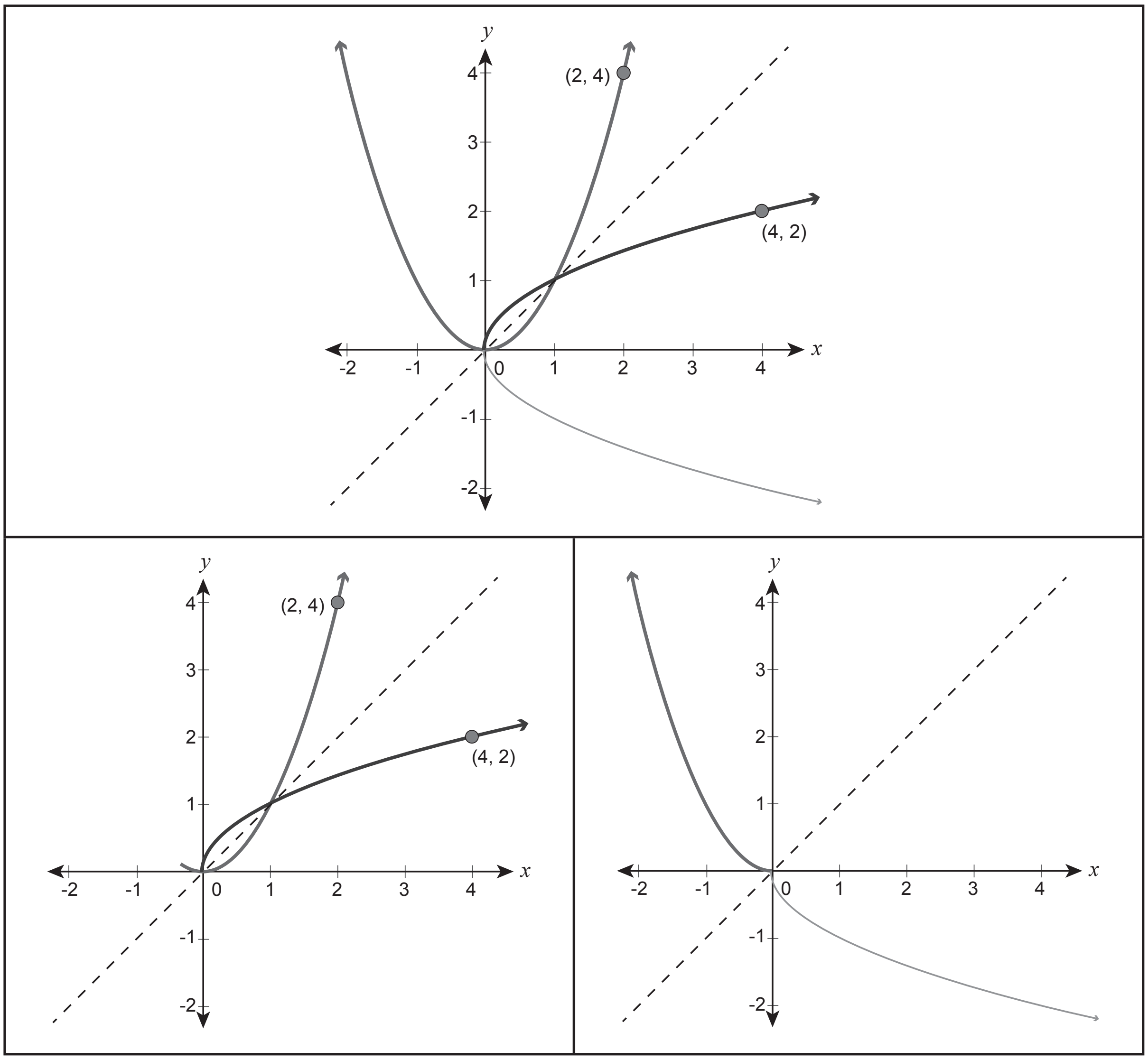
13. Tell learners that the original quadratic function's equation is: $y = x^2$.
    To find the inverse:

Interchange the x-values and y-values	$x = y^2$
Say: *Although this is the equation for the inverse of the function, it is not in standard form.* y *should be the subject of the formula. To do this we need to use the inverse operation of squaring – square rooting. Remember that a square root of a number could produce two possible solutions.*	$x = y^2$ $\pm\sqrt{x} = \sqrt{y^2}$ $y = \pm\sqrt{x}$

14. Ask: *Look at the diagram again of these two graphs. Is the inverse graph a function?*
    (No, for every possible x-value there are two possible y-values).

15. Tell learners that this is a very important concept. Say:
    Let us discuss what changes we can make to the original function so that when we find the inverse it would also be a function.

16. Use the original sketch and erase accordingly to change the parabola so that its inverse will be a function. Ensure both options are done by re-doing the sketch and erasing again.



17. As these adjustments are made, show learners that the inverse of the original is still a function: every x-value has only one possible y-value.

18. Once learners have understood visually what needs to be adjusted on the parabola for its inverse to also be a function, discuss it more formally.

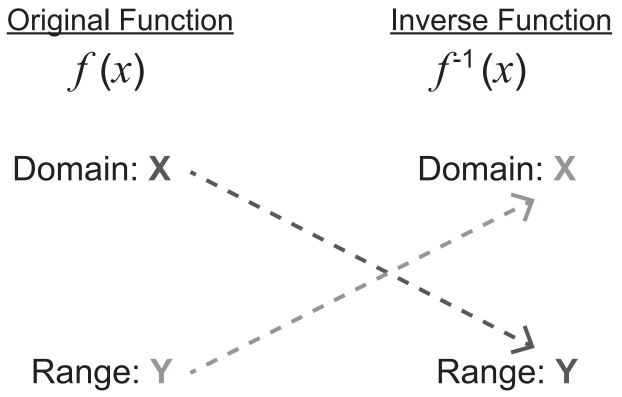
19. Ask: *What part of the parabola did I draw in the first diagram?*
    (The part that corresponded only with the positive values of x).

20. Ask: *What part of the parabola did I draw in the second diagram?*
    (The part that corresponded only with the negative values of x).

21. Say: *This means that the domain has been restricted.*
    To ensure that the inverse of a quadratic is still a function, the domain of the quadratic must be restricted. Tell learners to write this in their books.

22. Before doing some examples with learners, summarise all that has been said and ask learners to write it in their books. Encourage learners to ask questions as you discuss the summary.

An inverse function is a reflection of a function in the line $y = x$
Rule to follow to find the inverse of a given function: $(x\,;y) \rightarrow (y\,;x)$
$f(x) \rightarrow$ denotes a function
$f^{-1}(x) \rightarrow$ denotes the inverse of a function
The domain of a quadratic function (the parabola) would have to be restricted to make the inverse a function.

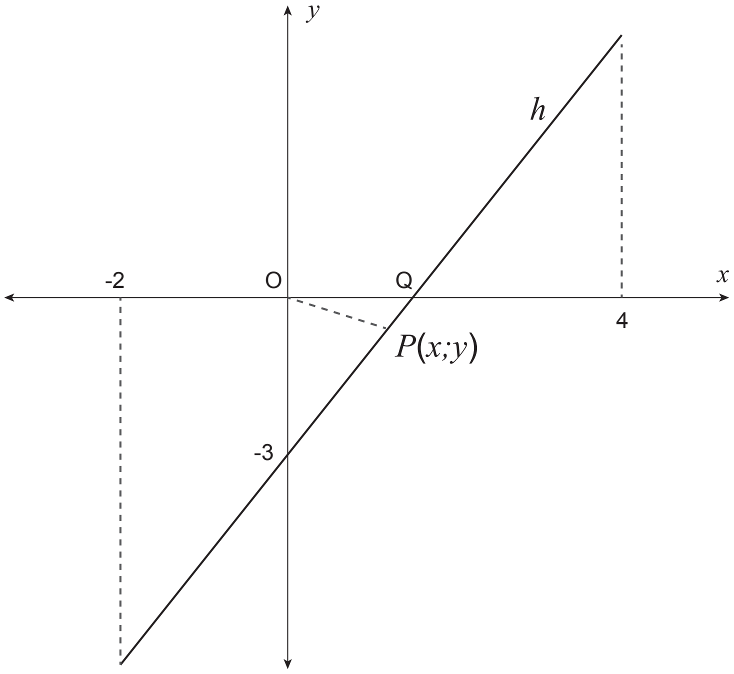


23. Do a fully worked example from a past paper with learners.
    Learners should write the example in their books, making notes as they do so.

Note that this question had another part to it using the point marked on the diagram $P(x\,;y)$. This part requires a knowledge of Calculus and has therefore been left out of the example.

Example:

Given $h(x) = 2x - 3$ for $-2 \leq x \leq 4$. The x-intercept of h is Q.



a) Determine the coordinates of Q.
b) Write down the domain of h^{-1}
c) Sketch the graph of h^{-1}, clearly indicating the y-intercept and the end points.
d) For which values of x, will $h(x) = h^{-1}(x)$?

Teaching notes:

a) This should be easy for learners. To find the x-intercept of any graph, make $y = 0$.
b) Point out that there is no need to find or draw the inverse function to answer this question. Remind learners that the domain of the inverse function is the same as the range of the function. Hence, they should find the range of the function given. This can be done by substituting the two end values of the domain to find the two end values of the range.
c) Remind learners that the y-intercept of the inverse function is the same value as the x-intercept of the function. This was found in a).

   The end points can be shown either using the answer found above or by using the domain of the original function and showing these values as the range of the inverse function.
d) Before finding the point of intersection of the function with its inverse function, the equation of the inverse function needs to be found. Note that the full point of intersection is not asked for.

Solutions:

a) Let $y = 0$

$2x - 3 = 0$

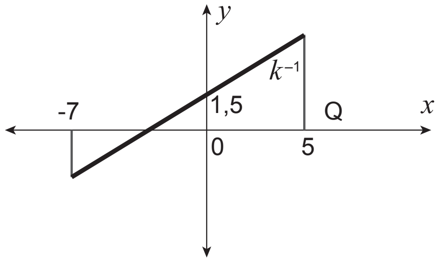
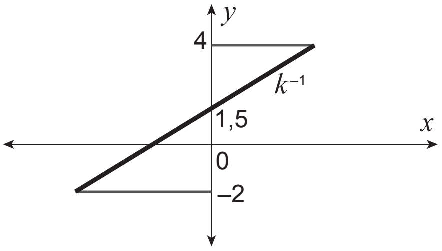
$x = \frac{3}{2} \quad \therefore Q\left(\frac{3}{2}; 0\right)$

b) $h(-2) = 2(-2) - 3$ and $h(4) = 2(4) - 3$

$h(-2) = -7$ $\qquad$ $h(4) = 5$

$\therefore$ domain of $h^{-1}: -7 \leq x \leq 5$

c)

Showing the endpoints using x-values	Showing the endpoints using y-values
	

d) $h(x) = 2x - 3$

Inverse: $x = 2y - 3$

$x - 3 = 2y$

$y = \frac{x}{2} - \frac{3}{2}$

$h^{-1}(x) = \frac{1}{2}x - \frac{3}{2}$

$h(x) = h^{-1}(x)$

$2x - 3 = \frac{1}{2}x - \frac{3}{2}$

$4x - 6 = x - 3$

$4x - x = -3 + 6$

$3x = 3$

$x = 1$

24. Ensure learners work through the exercises in the textbook. Correct on a regular basis – do not wait until all the exercises are complete. Rather correct and explain problems that arise each day.

ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=52tpYl2tTqk

https://www.youtube.com/watch?v=87Duk_ySbto

https://www.youtube.com/watch?v=34tP7UkbkV0

TERM 1, TOPIC 2, LESSON 4

REVISION OF EXPONENTIAL LAWS AND EXPONENTIAL FUNCTIONS

Suggested lesson duration: 1 hour

A POLICY AND OUTCOMES

CAPS Page Number	40, 41
Lesson Objectives By the end of the lesson, learners will have revised: ● exponential laws and definitions ● exponential functions.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson have the table ready to populate with the four laws and two definitions.
5. The table below provides references to this topic in Grade 12 textbooks. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**CLASSROOM MATHS**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
		1	48	Qu's	58	3.1 3.2	63	3.1 3.2	56 60		

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Exponents is a topic that many learners find difficult. Revising this section can only be beneficial. Take the time to do it thoroughly.

2. Exponential functions also need to be revised and practiced before the inverse of these functions can be covered. Although it was covered briefly in one of the examples in the revision lesson, it will be revised in more detail now.

DIRECT INSTRUCTION

1. Start the lesson by asking learners to tell you the four laws and two definitions of exponents.

2. Draw up a table as follows:

	With variables:	Basic example:	Explanation in words:
Laws:			
1			
2			
3			
4			
Definitions:			
1			
2			

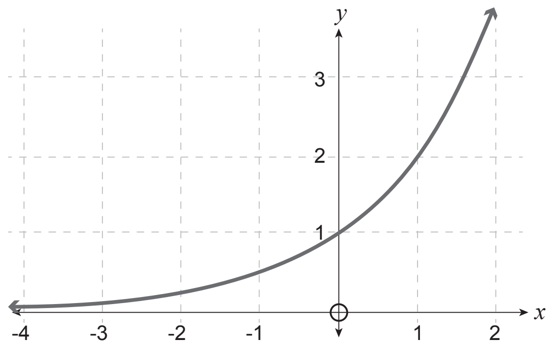
3. As you are given a law or a definition (be it in words or as an example), write it in the appropriate block and populate the other two areas for that rule.
   Below is an example of what the table should look like when complete. It does not need to be identical – use correct information from the learners.

	With variables:	Basic example:	Explanation in words:
Laws:			
1	$a^m \times a^n = a^{m+n}$	$2^3 \times 2^2 \times 2$ $= 2^{3+2+1}$ $= 2^6$	When multiplying powers with like bases, keep the bases the same and add the exponents.

2	$\frac{a^m}{a^n} = a^{m-n}$	$\frac{6x^6}{2x^2}$ $= 3x^4$	When dividing powers with like bases, keep the base and subtract the exponent. Divide coefficients (numbers) as per normal.
3	$(a^m)^n = a^{mn}$	$(3^5)^3$ $= 3^{15}$	When raising exponents to a power, keep the base and multiply the exponents.
4	$(ab)^m = a^m b^m$ $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$(2b^4)^3$ $2^3 b^{12}$	When more than one base is raised to an exponent, each base is raised to the exponent. When a fraction is raised to an exponent, the numerator and denominator must be raised to that exponent.
Definitions:			
1	$a^0 = 1$	$2^o = 1$ $2x^o = 2$	Any base raised to the power of zero is equal to 1. ($x \neq 0$ as 0^0 is undefined)
2	$a^{-a} = \frac{1}{a^a}$	$3^{-1} = \frac{1}{3}$ $3a^o b^{-2} = \frac{3}{b^2}$	A base raised to a negative exponent is equal to its reciprocal raised to the same positive exponent.

4. Tell learners that their exponent work needs to be good – exponents are part of many topics such as algebra, finance, probability and functions.
   Say: *Functions will be our focus today.*

5. Ask learners to draw a sketch of the function $y = 2^x$
   Say: *The sketch need not be accurate but mark any intercepts or asymptotes clearly.*



6. Discuss these aspects of exponential functions:
   - There is one asymptote. It is a horizontal line and therefore in the form $y = q$
   - The y-intercept will be (0 ; 1) providing there have been no vertical or horizontal shifts

- The domain of an exponential function is $x \in \mathrm{R}$
- The range is limited by the asymptote
- There is no axis of symmetry.

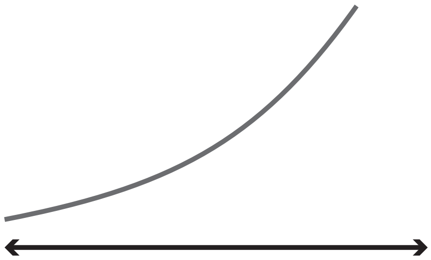
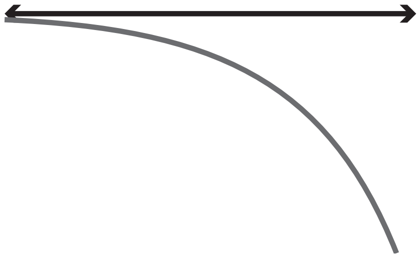
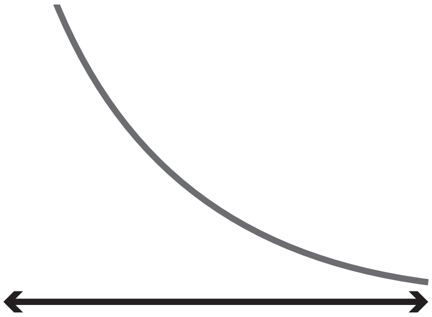
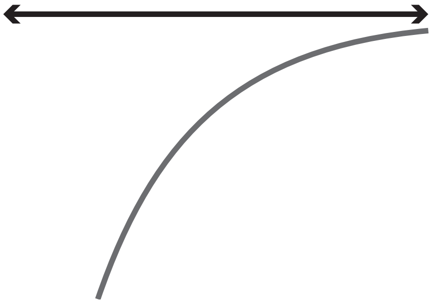
7. Revise the other aspects of exponential functions. Discuss the complete form of an exponential function and what each variable represents. Learners should write notes in their books.

horizontal shift

$$y = a.b^{x-p} + q$$

vertical shift (and the asymptote)

The effect of a and b can be summarised as follows:

	$a > 0$	$a < 0$
$b > 1$		
$0 < b < 1$		

8. Tell learners to write the summary in their books and to note that the horizontal line represents the asymptote.

9. Give learners a few examples to draw:

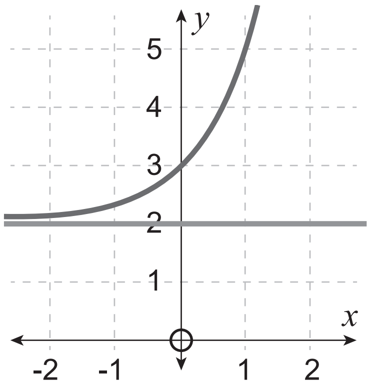
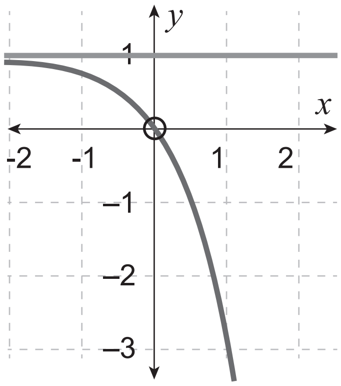
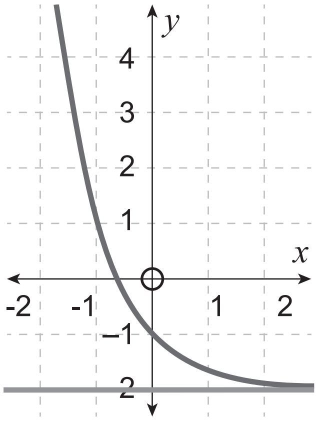
$y = 3^x + 2$	$y = (-1).4^x + 1$	$y = \left(\frac{1}{3}\right)^x - 2$

Tell learners to consider the following when drawing an exponential function:

- Decide if the function is increasing or decreasing and whether it is positive or negative
- Note that 'positive' means above the horizontal asymptote and 'negative' means below the horizontal asymptote.
- Find the equation of the asymptote and draw it in if it is not $y = 0$ (the x-axis)

- Find the y-intercept ($x = 0$)
- Use the values $x \in \{-1; 0; 1\}$ to find corresponding y-values to ensure accuracy.

10. Solutions:

$y = 3^x + 2$ The function is increasing (3 > 0) The function is positive ($a = 1$) Asymptote: $y = 2$ y-intercept: ($x = 0$) $y = 3^0 + 2$ $y = 1 + 2$ $y = 3$ (0;3)	
$y = (-1).4^x + 1$ The function is decreasing (4 > 0 but -1 < 0) The function is negative ($a = -1$) Asymptote: $y = 1$ y-intercept: ($x = 0$) $y = (-1).4^0 + 1$ $y = -1 + 1$ $y = 0$ (0;0)	
$y = \left(\frac{1}{3}\right)^x - 2$ The function is decreasing ($0 < \frac{1}{3} < 1$) The function is positive ($a = 1$) Asymptote: $y = -2$ y-intercept: ($x = 0$) $y = \left(\frac{1}{3}\right)^0 - 2$ $y = 1 - 2$ $y = -1$ (0;-1)	

11. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

12. Give learners an exercise to complete on their own.

13. Walk around the classroom as learners do the exercise. Support learners where necessary.

14. Once an exercise has been completed and corrected, and any problems have been addressed, tell learners to remind you how to find an inverse function.

15. Discuss what the inverse of an exponential function would look like.
16. Write the following two exponential functions on the board:

$y = 2^x$	$y = \left(\frac{1}{3}\right)^x$

17. Ask learners to perform what is necessary to start finding the inverse of these functions.

$y = 2^x$	$y = \left(\frac{1}{3}\right)^x$
Interchange the x- and y-values	
$y = 2^x$	$y = \left(\frac{1}{2}\right)^x$

18. Point out that, as yet, they have not learned how to make an exponent the subject of a formula. Hence, they cannot continue yet and write the inverse function in standard form.
19. Tell learners that this requires a knowledge of logarithms which will be covered in the next lesson. Ask learners to ensure they bring their calculators to the next lesson.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=A1wKTiBTsfk&t=423s

https://www.youtube.com/watch?v=dDnatQBMEu4

https://www.youtube.com/watch?v=ls78_2UBcdY

TERM 1, TOPIC 2, LESSON 5

LOGARITHMS

Suggested lesson duration: 2,5 hours

A POLICY AND OUTCOMES

CAPS Page Number	40, 41

Lesson Objectives

By the end of the lesson, learners should be able to:

- define a logarithm
- use logarithmic laws in simple form (they will be used again in the Finance section to find the number of payments).

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Have Resource 5 ready for use during lesson.
4. Write the lesson heading on the board before learners arrive.
5. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**CLASSROOM MATHS**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
5 6 7 8 9	54 56 57 58 59	2 3 4 5	50 52 53 54	1 2 3	61 63 65	3.3 3.4 3.5	68 71 72	3.3 3.4 3.5 3.6	62 65 67 70	2.6 2.7 2.8 2.9 2.16 2.17 2.18	74 76 78 82 100 102 108

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Some textbooks have many exercises on the laws of logarithms and logarithmic equations. Use your textbook at your discretion – it is best not to leave out an entire exercise, but you could choose certain questions within each exercise for learners to complete.
2. Manipulation involving the logarithmic laws will not be examined. However, learners still need to understand the concept well.

DIRECT INSTRUCTION

Introduction to and understanding logarithms

1. Start the lesson by asking: *How many 2s do we multiply to get 8?*
   (3)
   Write $2^3 = 8$ on the board and circle the exponent. Tell learners to write it in their books.
2. Say: *If we are trying to find the exponent, we use logarithms.*
   Write 'logarithms' on the board.
3. Tell learners that in the above example, the logarithm would be 3.
   Say: *Let's look how to write it:*

   $$\log_{2} 8 = 3$$

   Tell learners to write this and what follows in their books.
   This means: The number of twos we need to multiply to get 8 is 3.
   We read it as: log base 2 of 8 equals 3.
4. Tell learners to note then that the following represents the same concept:

   $$\underbrace{2 \times 2 \times 2}_{3} = 8 \quad \longleftrightarrow \quad \log_{\underset{\nwarrow\,\text{base}}{2}}[8] = 3$$

   Show learners that the base of a log is the base of an exponential statement.
   Say: *the base goes to the bottom in the log.*
5. Point to $\log_{2} 8 = 3$ again and discuss the three numbers used:
   - The base (2) is the number being multiplied
   - The answer (3) shows how often the base is being multiplied – this is the logarithm
   - The number we want to get to (8) by multiplying the base.

6. Do two more examples with learners. Explain what is being asked.

$\log_5 125$	$\log_8 64$
How many 5s need to be multiplied to get 125?	How many 8s need to be multiplied to get 64?
$\log_5 125 = 3$	$\log_8 64 = 2$

7. Give learners the following examples to do on their own. Tell learners not to use a calculator. They should ask themselves the appropriate question and write the answer. Once learners have had enough time to answer the questions discuss the solutions with them. For each question, ask learners to tell you the question they should have asked.

Example	Solution	Because:
$\log_3 81$	4	$3^4 = 81$
$\log_7 49$	2	$7^2 = 49$
$\log_9 81$	2	$9^2 = 81$
$\log_2 64$	6	$2^6 = 64$
$\log_2 1$	0	$2^0 = 1$
$\log_{10} 10\ 000$	4	$10^4 = 10\ 000$

8. Discuss the last two examples with learners.

Example	Discussion
$\log_2 1 = 0$	Say: *Any log question that must equal 1 will have an answer of zero. This is because any number to the exponent zero is equal to 1 (except 0 as that is undefined).* Ask learners to make up a few more log statements that would equal zero. Allow a few learners to come to the board and write them up for the class. Possible examples: $\log_5 1$; $\log_{10} 1$; $\log_{24} 1$; $\log_{90} 1$
$\log_{10} 10\,000 = 4$	Base 10 is considered a common logarithm. If a logarithm is written without a base, the base is 10. This question could have been written as: log 10 000 = 4. There is no need to write a base in the logarithm if it is 10. Tell learners to take out their calculators and find the 'log' button. Ask learners to press log 10 000 and confirm they get 4. Do a few more examples that give a whole number as an answer such as log 100 (2) or log 1 000 000 (6).

9. Use the previous information regarding base 10 and ask learners to use their calculators to find:

log 26	log 2000
Point out that so far, all the answers to logarithms seemed to have been whole numbers. Learners should note that decimals are also possible. Say: *We will explore this further in the next lesson when we draw logarithmic functions, and in finance when the number of payments (also represented by an exponent) is not a whole number.*	
log 26 = 1,415	log 2000 = 3,301

10. Ask: Is it possible to have negative logarithms?
    (After giving learners chance to give their opinion point out the following:
    - Although logarithms deal with multiplying, a negative logarithm just means we deal with dividing
    - A negative logarithm means how many times to divide by a number).

11. For example:

$$\log_2 0{,}5$$

Say: *Because* 1 ÷ 2 = 0,5 *the logarithm above is equal to* –1.

If the number that we need to get to is smaller than the base, we need to divide, which means there will be a negative answer.

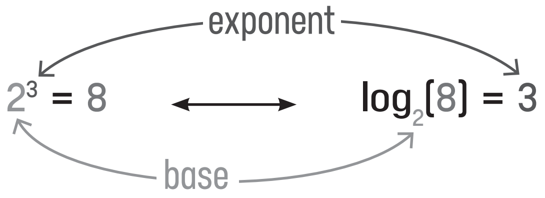
Note: There is a table available in the Resource Pack that makes this concept clearer.
See Resource 5

Converting between exponential statements and logarithms

12. Learners must be able to change between exponents and logarithms.
    Write the following on the board:

$$2^3 = 8$$

Say: *We have seen this as a logarithm before so let's use it to note where each number goes in the logarithm statement.*



Tell learners to note:
- The exponent is the answer to a logarithm
- The base of the power is the base of the logarithm – it goes to the bottom.

13. Write the general rule on the board:

$$\text{If } y = b^x \text{ then } x = \log_b y$$

Tell learners to write the general rule in their books.

14. Give the following exponential statements for learners to change into logarithms. Once they have had enough time to complete them, discuss and correct them on the board.

Exponential form	Logarithmic form
$5^4 = 625$	$\log_5 625 = 4$
$7^3 = 343$	$\log_7 343 = 3$
$64 = 4^3$	$\log_4 64 = 3$
$59049 = 9^5$	$\log_9 59049 = 5$

15. Give the following logarithms to learners to change into exponential form. Once they have had enough time to complete them, discuss and correct them on the board.

Logarithmic form	Exponential form
$\log_2 32 = 5$	$2^5 = 32$
$\log_6 279936 = 7$	$6^7 = 279936$
$\log_{15} 3375 = 3$	$15^3 = 3375$
$\log 1000 = 3$	$10^3 = 1000$

16. As a final practice exercise, learners need to be able to change exponential equations into logarithms in order to solve for the exponent. Do two examples with learners:

Exponential equation:	Change to logarithm	Use the calculator to solve:
$12^x = 144$	$x = \log_{12} 1728$ $x = 2$	Tell learners to find this button on their calculator  Tell learners to press this button then fill the numbers in and press equal. The first answer is a whole number and an easy one so that learners can see that it is correct. The second answer involves a decimal.
$2^x = 42$	$x = \log_2 42$ $x = 5{,}392$	

17. Give learners the following examples to try on their own. Once they have had enough time to complete the questions, discuss and correct them on the board.

Equation:	Solution
$2^x = 64$	$x = \log_2 64$ $x = 6$
$6^a = 120$	$a = \log_6 120$ $a = 2{,}672$
$3^y = 145{,}62$	$y = \log_3 145{,}62$ $y = 4{,}534$

Logarithm laws

Note: Learners are not required to know these laws. However, being exposed to the laws gives learners a better understanding of logarithms and their close connection to exponents.

18. Draw up a table as follows:

Exponent law	Logarithm law	Example:
$a^m \times a^n = a^{m+n}$	$\log (ab) = \log a + \log b$	$\log_3 27 + \log_3 9$ $= \log_3 (27 \times 9)$ $= \log_3 81$ $= \log_3 3^4$ $= 4 \log_3 3$ $= 4(1) = 4$
$\frac{a^m}{a^n} = a^{m-n}$	$\log \left(\frac{a}{b}\right) = \log a - \log b$	$\log_3 27 - \log_3 9$ $= \log_3 \left(\frac{27}{9}\right)$ $= \log_3 3$ $= 1$
$(a^m)^n = a^{mn}$	$\log a^n = n \log a$	$\log_2 8 = \log_2 2^3$ $= 3 \log_2 2$ $= 3(1)$ $= 3$
$a^1 = a$	$\log_a a = 1$	
$a^0 = 1$	$\log_a 1 = 0$ or $\log 1 = 0$	$\log_{12} 1$ $= 0$

19. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

20. Give learners an exercise to complete with a partner.

21. Walk around the classroom as learners do the exercise. Support learners where necessary.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=poDYtJ86jM0

https://www.youtube.com/watch?v=p99-In_bn1M

https://www.youtube.com/watch?v=ViFjyCNHKT4

TERM 1, TOPIC 2, LESSON 6

LOGARITHMIC FUNCTIONS

Suggested lesson duration: 2 hourS

POLICY AND OUTCOMES

A

CAPS Page Number	40, 41

Lesson Objectives

By the end of the lesson, learners should be able to:

- recognise and sketch logarithmic functions
- answer questions related to logarithmic functions.

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Have Resource 6 ready for use during lesson.
4. Write the lesson heading on the board before learners arrive.
5. Write work on the chalkboard before the learners arrive. For this lesson draw a Cartesian plane.
6. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**CLASSROOM MATHS**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
4	51	6	55	4	69	3.6	76	3.7	76	2.10 2.11	87 89

C

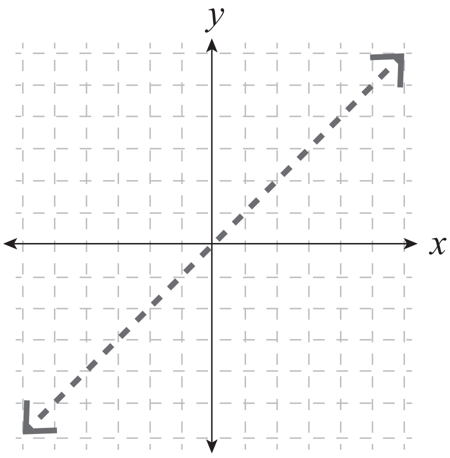
CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Logarithmic functions are key to understanding the inverse of exponential functions.
2. Learners must be able to answer questions regarding domain and range of functions and their inverses.

DIRECT INSTRUCTION

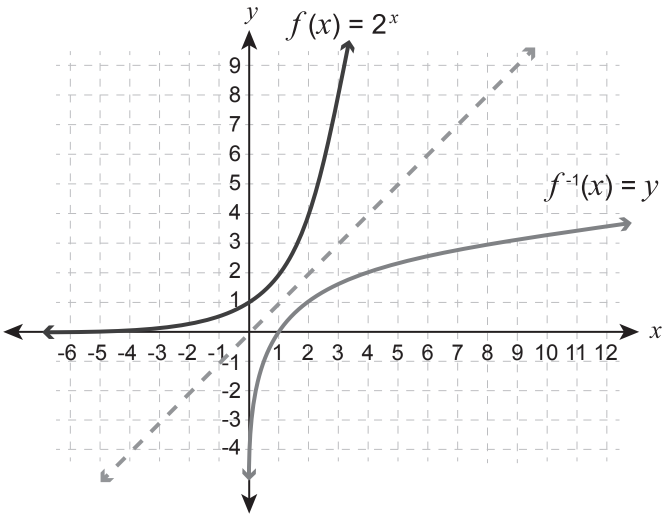
1. Ask: *What is an inverse function?*
   (A reflection of the function in the line $y = x$).
2. Tell learners to draw a Cartesian plane and the line $y = x$ onto it.



3. Once learners have this ready, ask them to sketch the function $y=2^x$ on the same Cartesian plane, marking the y-intercept clearly.

Learners should be able to do this quickly and efficiently.

4. Ask learners to draw in the inverse function. Remind them that if they folded their paper along the $y = x$ line the two graphs should land on top od one another. As they are drawing, remind learners that all x-values become y-values and all y-values become x-values. Tell them they should focus on the asymptote of the exponential function and think about where the asymptote will be of the inverse function.
5. Once learners have drawn their own graphs, draw the following on the board to confirm that they have drawn their graphs correctly.



6. The graph need not be accurate. It just needs to show the y-intercept of the exponential graph and the x-intercept of the inverse function. It should also be clear that the line $y = 0$ (the x-axis) is the asymptote of the exponential function and that the line $x = 0$ (the y-axis) is the asymptote of the inverse function. Discuss these points with learners.

7. Ask: *Is the exponential function increasing or decreasing?*
   (Increasing).
   Ask: *Is the inverse function increasing or decreasing?*
   (Increasing).

8. Remind learners how to write the inverse of a function: $f^{-1}(x)$ and label the graph as above.

9. Write the equation of the exponential function on the board:

$$y = 2^x$$

10. Find the equation of the inverse function with learners:

$$x = 2^y$$

    Use logarithms to make y the subject of the formula.

$$y = \log^2 x$$

11. Tell learners to write this equation next to the log graph they drew.

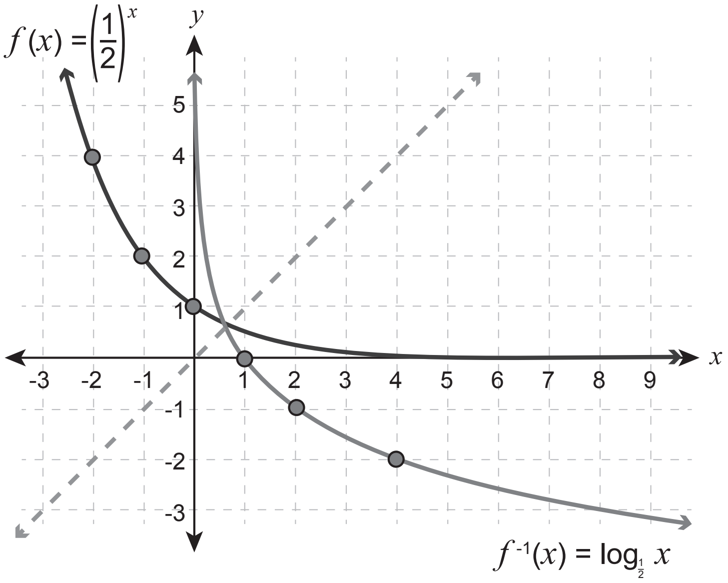
12. Ask learners to draw another Cartesian plane with the line $y = x$ drawn in, then ask them to draw the function $y = \left(\frac{1}{2}\right)^x$.

> Some learners may need to be reminded that because the base lies between 0 and 1, the function will decrease.

13. Once learners have drawn the exponential function ask them to draw the inverse of the function. Remind them to consider the fact that it must be reflected in the line $y = x$ and also to consider the intercepts and the asymptotes.

14. Once learners have completed their sketches, draw them on the board for learners to check and correct their work if necessary.

15. As you are drawing the graphs, discuss the intercepts and the asymptotes. Ensure that learners can see the connections between the exponential function and its inverse.



16. If learners didn't write the equation of each function on the diagram, tell them to do so now.

17. Use the diagram to discuss other aspects of functions and their inverses before doing a fully worked example with learners.

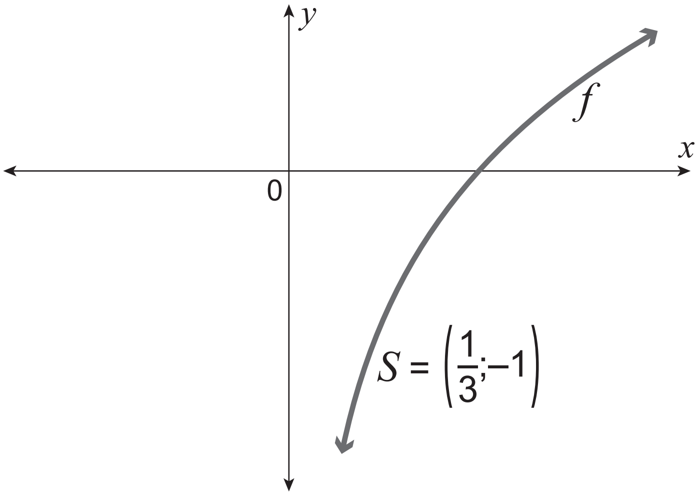
18. Ask each question and give learners a minute or so to answer before discussing and correcting it with them

Question	Solution	Notes
Write down the range of $f(x) = \left(\frac{1}{2}\right)^x$	$y \in (0;\infty)$ or $y > 0$	Point out that once the range of the function has been determined, it should be clear that the domain of the inverse function will be made up of the same set of numbers.
Write down the domain of $f^{-1}(x) = \log_{\frac{1}{2}} x$	$x \in (0;\infty)$ or $x > 0$	
Write down the values of x for which $f(x) \leq 4$	$x \in (-2;\infty)$ or $x > -2$	These values can be read directly from the graph or calculated algebraically.
Write down the values of x for which $f^{-1}(x) > -2$	$x \in (0;4)$ or $0 < x < 4$	

19. Tell learners that you are going to do one fully worked example from a past paper. Learners should write the example in their books and make notes as they do so.

Example:

Given: $f(x) = \log_a x$ where $a > 0$. $S\left(\frac{1}{3};1\right)$ is a point on the graph of f.



a) Prove that $a = 3$

b) Write down the equation of h, the inverse of f, in the form $y = ...$

c) If $g(x) = -f(x)$, determine the equation of g.

d) Write down the domain of g.

e) Determine the values of x for which $f(x) \geq -3$

Teaching notes:

a) Remind learners that when they are asked to prove something they may not use the given information. They should rather see the question as: Find the value of a and use what has been given as a check. The point S needs to be used in the given equation and then a can be solved for. Knowledge of exponent and logarithm laws will be used to change the logarithm statement into an exponential statement. From this point, it will be possible to solve for a.

b) Ensure learners remember the key points to finding the inverse of a given function. Change the x and y (but remind them why! – it is a reflection on the line $y = x$). Then an understanding of logarithms and functions will be required to make y the subject of the formula.

c) Ask: *What transformation has happened to a function if it has become a negative function?*

(It is a reflection in the line $y = 0$ or the x-axis).

Ask: *What rule will make this transformation?*

$(x;y) \rightarrow (x;-y)$.

d) Sketch the function g onto the Cartesian plane with $f(x)$ so that learners can see the domain of the function. Point out that the domain of g should be the same as the domain of f as it has been reflected in the x-axis and therefore the same x-values are used.

e) Point out that learners will first need to find the corresponding x-value when $y = –3$. Once this has been found it would be a good idea to show this point on the diagram to assist learners in seeing visually where the function is equal to or larger than –3. Once it has been added to the sketch, draw the line $y = –3$ and highlight the part of the function that lies above it. Then ask learners to re-focus their attention to what the x-values are for that part of the graph.

Solution:

a) $f(x) = \log_a x$

$-1 = \log_a \frac{1}{3}$

$\therefore a^{-1} = \frac{1}{3}$

$a = \left(\frac{1}{3}\right)^{-1}$

$a = 3$

b) $f(x) = \log_3 x$

$x = \log_3 y$

$y = 3^x$

c) $f(x) = \log_3 x$

$\therefore -f(x) = -\log_3 x$

$\therefore g(x) = -\log_3 x$ (other options: $g(x) = \log_3 \frac{1}{x}$; $g(x) = \log_{\frac{1}{3}} x$

d) $x > 0$ or $x \in (0;\infty)$

e) $\log_3 x = -3$

$x = 3^{-3}$

$x = \frac{1}{27}$

$\therefore x \geq \frac{1}{27}$

20. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

21. Give learners an exercise to complete on their own.

22. Walk around the classroom as learners do the exercise. Support learners where necessary.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=AAW7WRFBKdw

https://www.youtube.com/watch?v=AAW7WRFBKdw

https://www.youtube.com/watch?v=q9DhlR43P7A

TERM 1, TOPIC 2, LESSON 7

REVISION AND CONSOLIDATION

Suggested lesson duration: 2 hours

POLICY AND OUTCOMES

A

CAPS Page Number	40, 41
Lesson Objectives By the end of the lesson, learners will have revised: ● the concepts of Functions and their inverses.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		CLASSROOM MATHS		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
Rev S Ch	60 61	Rev	56	5 Qu's	73 74	Rev	78	3.8 3.9	78 80	2.12 2.13 2.14 2.15	91 94 97 99

C CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Ask learners to summarise what they have learned in this section. Point out issues that you know are important as well as problems that you encountered from your own learners during the course of this topic.
2. If learners want you to explain a concept again, do that now.
3. As you work through any explanations with the learners, it is important to frequently talk about as many concepts as possible. For example, use the words function, domain, range, increasing, decreasing and inverse function.

DIRECT INSTRUCTION

1. Ask learners to do the revision exercise from their textbook. If you have any past paper questions for them, these can also be done now. This allows learners the opportunity to become more familiar with what will be expected of them in the final examination.
2. Walk around the classroom as learners do the exercise. Support learners where necessary.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=ZPLjsRoKcJg&t=0s&index=13&list=PLOaNAKtW5HLT_jez2u6y-bipAJha8xYZ0X

https://www.youtube.com/watch?v=gvdLrGe2AWI

https://www.youtube.com/watch?v=EFPoY3ZWqnY

Term 1, Topic 3: Topic Overview

FINANCE, GROWTH AND DECAY

A. TOPIC OVERVIEW

- This topic runs for two weeks (9 hours).
- It is presented over six lessons.
- The lessons have been divided according to sub-topics, not according to one school lesson. An approximate time has been allocated to each lesson (which will total 9 hours). For example, one lesson in this topic could take two school lessons. Plan according to your school's timetable.
- Finance, Growth and Decay counts 10% of the final Paper 1 examination.
- This section covers future and present value annuities as well as an aim to make learners more aware of the options for both savings and loans and to make informed decisions.
- It is important that all learners understand financial matters. This is learned and is not taught by all parents – making the teacher's role at school even more important. Financial mathematics is a topic that is particularly applicable to everyday life.

Breakdown of topic into 6 lessons:

	Lesson title	**Suggested time (hours)**		**Lesson title**	**Suggested time (hours)**
1	Revision	1	4	Finding the time period	1,5
2	Future value annuities	2,5	5	Analysing investments and loans Pyramid schemes	0,5
3	Present value annuities	2,5	6	Revision and consolidation	1

B SEQUENTIAL TABLE

GRADE 11 and earlier	GRADE 12
LOOKING BACK	**LOOKING FORWARD**
● Use the simple and compound growth formulae to solve problems including hire purchase, inflation and population growth ● The effect of different periods of compounding growth and decay ● Use simple and compound decay formulae to solve problems, including: - Straight line depreciation - Reducing balance depreciation - Nominal and effective interest rate	● Solve problems involving present and future value annuities ● Calculate the value of n (time period) using logarithms ● Critically analyse different loan options

C WHAT THE NSC DIAGNOSTIC REPORTS TELL US

According to **NSC Diagnostic Reports** there are a number of issues pertaining to Finance, Growth and Decay.

These include:

- a lack of understanding of the term depreciation
- basic algebraic skills, such as multiplication and exponential laws, need to be taught properly then reinforced throughout the FET phase
- learners should be encouraged to check whether their solutions seem ‘reasonable’
- when to use the present and future value formulae
- what situations affect n (and which situations do not affect it)
- deferred payments
- calculator skills.

It is important that you keep these issues in mind when teaching this section.

It is important to make Financial mathematics real for the learners.
Where possible, use examples from daily life which make sense to learners. Finance needs to be taught with insight and not merely as substitution into a formula.

ASSESSMENT OF THE TOPIC

D

- Two tests with memoranda are provided in the Resource Pack. The tests are aligned to CAPS in every respect, including the four cognitive levels as required by CAPS (page 53).
- The questions usually take the form of real life situations that require, for example, calculations regarding regular payments for both saving and repaying loans.
- Monitor each learner's progress to assess (informally) their grasp of the concepts. This information can form the basis of feedback to the learners and will provide you valuable information regarding support and interventions required.

MATHEMATICAL VOCABULARY

E

Be sure to teach the following vocabulary at the appropriate place in the topic:

Term	**Explanation**
principal amount	The initial or capital sum of money Can represent a borrowed or invested amount of money
profit	Money made when income exceeds (is bigger than) expenditure (You made more money than you spent)
loss	Money lost when expenditure exceeds income. (You spent more money than you made)
discount	Paying less than the usual price A percentage of the original price is taken off
budget	A plan to manage money
loan	Borrowing or lending money
interest	Money paid for borrowing money Money paid for the use of other money Money received for lending money
interest rate	The percentage of interest charged (loan) or received (investment). Rate is often quoted per annum.

simple interest	Interest on a loan, calculated on a yearly basis Interest on a loan amount is charged taking only the principal amount into consideration
instalment	A sum of money due as one of several equal payments for something, spread over an agreed period of time
exchange rate	The rate of one country's money against another country's money
VAT	Value Added Tax is paid on goods or services In South Africa VAT is 15% of the price
hire purchase	When an item is purchased but only a deposit is paid and then the item is paid off monthly over a specified period of time The interest charged is always simple interest The goods bought are being hired until the payments are complete
deposit	A sum of money paid as a first instalment on an item with the understanding that the balance will be paid at a later stage
compound interest	Interest is calculated not only on the principal amount but also includes any accumulated interest that has been added at certain time intervals
inflation	A general increase in prices and fall in the purchasing value of money. Inflation is always calculated using compound interest
fixed deposit	A single deposit invested for a certain period of time at a fixed interest rate
depreciation	A reduction in the value of an asset over time, due in particular to wear and tear
nominal interest	The interest rate before taking inflation into account Nominal can also refer to the advertised or stated interest rate on a loan, without taking into account any fees or compounding of interest
effective interest	The effective annual interest rate is the interest rate that is actually earned or paid on an investment, loan or other financial product due to the result of compounding over a given time period
annuity	A series of payments at fixed intervals

future value annuity	An annuity in which the aim of the payments is to save for the future The future value of the annuity will be the total amount saved
present value annuity	An annuity in which the aim of the payments is to pay back a loan The present value of the annuity is the total amount borrowed
deferred payment	Temporary postponement of the payment of an outstanding bill or debt, usually involving repayment by instalments
sinking fund	Fund formed by periodically setting aside money for the gradual repayment to replace a wasting asset
pyramid scheme	A form of investment in which each paying participant recruits further participants, with returns being given to early participants using money contributed by later participants Illegal in some countries

TERM 1, TOPIC 3, LESSON 1

REVISION

Suggested lesson duration: 1 hour

A POLICY AND OUTCOMES

CAPS Page Number	42

Lesson Objectives

By the end of the lesson, learners will have revised:

- nominal and effective interest rates
- depreciation
- different compounding periods.

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. If there isn't a revision exercise in the textbook that you use, either use the revision exercise at the end of a Grade 11 textbook or items from a Grade 11 test on Finance.
5. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		CLASSROOM MATHS		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
Rev	62			Qu's	77	4.1	85	4.1	85		

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. This lesson will take the form of going through questions from two Grade 11 final examinations. As you work through the solutions, remind or re-teach learners all the concepts covered in previous years.
2. Although Finance is not a major part of the final examination, it is one of the topics that will be useful to all learners after school. Everyone needs some knowledge of financial matters in their own lives – particularly once they start working and earning money.

DIRECT INSTRUCTION

1. Ask what learners remember learning in financial mathematics in Grade 10 and Grade 11. As topics are given, write them on the board. Tell learners to write the list in their own books as well. Ensure the following items are on the list.
    - Simple interest
    - Compound interest
    - Exchange rates
    - Hire purchase
    - Inflation
    - Depreciation
    - Nominal and effective interest rates

    Ask learners questions to assess what they remember from last year.
    Tell learners to take notes when you sum up (or add to) what they tell you.

2. Ask: *What is the formula for simple interest?*

    $A = P(1 + n.i)$.

    Ask: *What do the* A, P, n *and* i *represent?*
    (Accumulated or final amount; Principal or starting amount; number of years; interest rate).

3. Ask: *What is the formula for compound interest?*

    $A = P(1 + i)^n$

    What do the A, P, n *and* i *represent?*
    (Accumulated or final amount; Principal or starting amount; number of times interest will be calculated and added; interest rate).

4. Ask: *what does it mean to buy an item on hire purchase?*
   (Pay a deposit, pay monthly instalments over a certain period of time usually at a high interest rate).
   Ask: *What type of interest is used for hire purchase?*
   (Simple).

5. Ask: *what does the term inflation mean?*
   (A general increase in prices and fall in the purchasing value of money).
   Ask: *What type of interest is used for calculating the rate of inflation?*
   (Compound).
   Ask: *Why?*
   (Because the new price each year needs to be taken into account; therefore it is 'interest on interest').

6. Ask: *What is depreciation?*
   (A reduction in the value of an asset over time).

7. Ask: *What is the difference between the nominal interest rate and the effective interest rate?*
   (The nominal rate is the rate given by the bank at the start of an investment or loan.
   The effective rate is the actual annual rate earned (on an investment) due to the compounding period being more often than once a year).

Learners should now have a summary of the financial mathematics they should already know.

8. Spend the rest of the lesson doing some questions from a previous Grade 11 examination. Encourage learners to help as you answer the questions; and/or should ask questions if they do not understand something.

Example

1. A tractor bought for R120 000 depreciates to R11 090,41 after 12 years by using the reducing balance method. Calculate the rate of depreciation per annum. (The rate was fixed over the 12 years).
2. Calculate the effective interest rate if interest is 9,8% p.a., compounded monthly.
3. Mrs Pillay invested R80 000 in an account which offers the following:
   - 7,5%p.a. compounded quarterly, for the first 4 years, and thereafter,
   - 9,2%p.a. compounded monthly, for the next 3 years.

   Calculate the total amount of money that will be in the account at the end of 7 years if no further transactions happen on the account.

4. Exactly 8 years ago Tashil invested R30 000 in an account earning 6,5%p.a., compounded monthly.
   a) How much will he receive if he withdrew his money today?
   b) Tashil withdrew R10 000 three years after making the initial deposit and re-invested R10 000 five years after making the initial deposit.
   Calculate the difference between the final amount Tashil will now receive after eight years and the amount he would have received had there not been any transactions on the account after the initial deposit.

NSC 2014

Teaching notes:

1. Ask: *What is the name of the other method of depreciation?*
   (Straight-line balance).
   Ask: *what is the difference between the two methods of depreciation?*
   (Straight-line is when the item will eventually be worth nothing.
   Reducing-balance is when the item is worth less and less as time goes by but will never reach zero).
   Ask learners to write the formula for reducing-balance depreciation and fill in the known values. Solve for i.
   Remind learners to never round off until the final step – answers should be kept in the calculator to ensure accuracy. Also remind learners that when they have found i, they still need to multiply by 100 to change it into a percentage.
2. Remind learners of the difference between effective and nominal interest rates.
   Nominal is the rate quoted by the bank. If this rate is compounded anything other than yearly, then effectively a savings would earn a bit more than the percentage quoted due to the accumulation over the time period. The effective interest rate is the rate that is really earned for one year.
   If a question asked to use the effective rate in a question, remember that it is an annual rate and the compounding periods such as monthly is no longer used.
   Whenever effective rate is found, remember to deal with one year only.
   Ask learners to write the formula down and fill in the given values then solve for i_{eff}.
3. A timeline could be useful in this case but, as there is only one change during the time of the investment, learners should be able to do the calculation without it. Remind learners that it can be done all at the same time by first ‘growing’ the money over the first compounding period and then use another bracket to grow it to the end.
   Tell learners to write down the formula to be used and fill in the values.
4. a) This is a straightforward compound interest question.
   b) A timeline will be very useful in this case. More than one change takes place during the time of the investment. Draw a timeline with learners.
   Remind them to find the difference. Between the two answers to answer the question.

Solutions

1.
$$A = P(1 - i)^n$$
$$11090{,}41 = 120000(1 - i)^{12}$$
$$\frac{11090{,}41}{120000} = \frac{120000(1 - i)^{12}}{120000}$$
$$\sqrt[12]{\frac{11090{,}41}{120000}} = \sqrt[12]{(1 - i)^{12}}$$
$$\sqrt[12]{\frac{11090{,}41}{120000}} = 1 - i$$
$$\therefore \quad i = 1 - \sqrt[12]{\frac{11090{,}41}{120000}}$$
$$\therefore \quad i = 0{,}1799\ldots$$
$\therefore$ the rate of depreciation is 18%

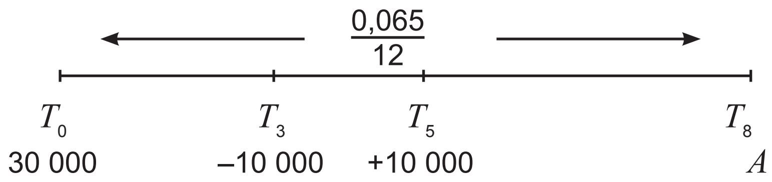
2.
$$i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n - 1$$
$$i_{eff} = \left(1 + \frac{0{,}098}{n}\right)^{12} - 1$$
$$i_{eff} = 0{,}102523\ldots.$$
$$\therefore \quad i_{eff} = 10{,}25\%$$

3. $80000\left(1 + \frac{0{,}075}{4}\right)^{16}\left(1 + \frac{0{,}092}{12}\right)^{36}$

= R141768,60

4. a)
$$A = P(1 + i)^n$$
$$= 30000\left(1 + \frac{0{,}065}{12}\right)^{8 \times 12}$$
$$= \text{R}50390{,}07$$

b)



$$A = 30000\left(1 + \frac{0{,}065}{12}\right)^{8 \times 12} - 10000\left(1 + \frac{0{,}065}{12}\right)^{5 \times 12} + 10000\left(1 + \frac{0{,}065}{12}\right)^{3 \times 12}$$
$$= \text{R}48708{,}61$$
$$\therefore \text{ difference} = \text{R}48708{,}61 - 50390{,}07$$
$$= -\text{R}1681{,}46$$

9. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

10. Give learners an exercise to complete on their own.

11. Walk around the classroom as learners do the exercise. Support learners where necessary.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=asMmXiFU_nc

TERM 1, TOPIC 3, LESSON 2

FUTURE VALUE ANNUITIES

Suggested lesson duration: 2,5 hours

A POLICY AND OUTCOMES

CAPS Page Number	42

Lesson Objectives

By the end of the lesson, learners should be able to:

- calculate the future value of annuity given monthly payment, interest and time period
- calculate the monthly payment using the future-value annuity formula.

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Have Resource 7 ready for use in this lesson.
4. Write the lesson heading on the board before learners arrive.
5. Write work on the chalkboard before the learners arrive. For this lesson have the future value formula ready.
6. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**CLASSROOM MATHS**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
2	74	1	61	1	81	4.2	88	4.2	90	3.2	114
3	80	3	68	6	93	4.3	92	4.3	97	3.3	117
8	99			7	95	4.5	100				

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Future value annuities is a new concept to learners. Finance is an important section – whether learners plan on continuing with mathematics or not after school, they will still need a good knowledge of financial mathematics.
2. Annuities are part of everyday life for most people. Tell learners that understanding annuities will assist them once they start earning their own money and want to save.

DIRECT INSTRUCTION

1. Start the lesson by explaining what an annuity is. This is probably a new word for learners.
2. Say: *An annuity is essentially a series of fixed payments. These payment s are made on a regular basis – the most common being monthly but they could be quarterly or annually too. Today we are going to be looking at the type of annuity where savings is the aim. This is called a Future value annuity.*
3. Point out the formula for a future value annuity on the board. Label and discuss each variable.

Regular payment

number of payments

$$F_v = \frac{x\,[(1 + i)^n - 1]}{i}$$

Future value

interest

4. Point out that there are four variables – at least three variables need to be known to find the unknown variable.
5. Say: *Finding the value of n requires the use of logarithms which will be covered in a later lesson once both future value and present value have been covered.*

> Do the following examples in which the future value and the monthly instalment is found. To give learners an opportunity to practice using the formula, values only are given instead of a word problem as will normally be the case.

6. Say: *I am going to do two examples with you now. Write them in your book, making notes as you do so.*

If x = R5000, i = 6% p.a. compounded monthly and the time period is 3 years, find the future value (F_v) of the annuity.

Tell learners they should always write the formula down then state all the known variables.

Solution

$$F_v = \frac{x[(1+i)^n - 1]}{i}$$

x = R5000 $i = \frac{0,06}{12}$ $n = 3 \times 12 = 36$

$$F_v = \frac{5000\left[\left(1+\frac{0,06}{12}\right)^{36} - 1\right]}{\frac{0,06}{12}}$$

$$F_v = \text{R}196\ 680,52$$

The future value of the investment is R196 680,52

If the future value of an annuity is R82 955,64 after 5 years and the interest rate was 7% p.a. compounded quarterly, find the quarterly instalment.

Remind learners that they should not round off until the final answer.
They should keep values in their calculator as they proceed with the calculation.

Solution:

$$F_v = \frac{x[(1+i)^n - 1]}{i}$$

$F_v = 82\ 955,64$ $i = \frac{0,07}{4}$ $n = 5 \times 4 = 20$

$$82955,64 = \frac{x\left[\left(1+\frac{0,07}{4}\right)^{20} - 1\right]}{\frac{0,07}{4}}$$

$$\frac{0,07}{4} \times 82955,64 = \frac{x\left[\left(1+\frac{0,07}{4}\right)^{20} - 1\right]}{\frac{0,07}{4}} \times \frac{0,07}{4}$$

$$1451.7237 = x\left[\left(1+\frac{0,07}{4}\right)^{20} - 1\right]$$

$$\frac{1451.7237}{\left(1+\frac{0,07}{4}\right)^{20} - 1} = \frac{x\left[\left(1+\frac{0,07}{4}\right)^{20} - 1\right]}{\left(1+\frac{0,07}{4}\right)^{20} - 1}$$

$$x = 3500$$

The quarterly instalment is R3 500.

7. Ask if there are any questions.

8. Discuss the formula again with learners. Say: *This formula works under the following circumstances: a person wants to start saving, once the details have been finalised with the bank it is the norm to make the first payment one month LATER. If a question mentions that*

a person starts paying one month after the start of the annuity, that is the way it works most of the time and the formula will simply be used as it is.

Do two examples where this basic use of the formula is the case.

9. Say: *I am going to do two examples with you where the more common case is used. After that, I will explain what to do when the normal situation is not the case.*
Write the examples in your book and make notes as you do so.

A small business owner sets up a sinking fund so that he can replace his computer, copier and printer in a few years' time. The first payment of R850 is made one month after the fund is set up. The final payment is made 6 years later. The interest rate offered is 7,2% p.a. compounded monthly. How much money will be in the fund at the end of 6 years?

Ask: *What is a sinking fund?*
(A fund formed to set aside money to replace assets that will depreciate over time).
Say: *An examination question could easily use the idea of inflation, depreciation and sinking funds to ask a few questions which would involve using the future value formula at some stage in the situation.*

Solution

$$F_v = \frac{x[(1+i)^n - 1]}{i}$$

x = R850 $\quad i = \frac{0,072}{12} \quad n = 6 \times 12 = 72$

$$F_v = \frac{850\left(1+\frac{0,072}{12}\right)^{72} - 1}{\frac{0,072}{12}}$$

$$F_v = 76265,98$$

The future value of the investment is R76 265,98

Lerato wants to pay cash for a car in 3 years' time. Taking inflation into account, she has calculated that the cost of the car will cost R180 000 by then. The bank offers her an interest rate of 8,4% p.a. compounded monthly.
What will her monthly instalment need to be for her to reach her goal?

Solution

$$F_v = \frac{x[(1+i)^n - 1]}{i}$$

$F_v = 180000 \quad i = \frac{0,084}{12} \quad n = 3 \times 12 = 36$

$$180\,000 = \frac{x\left[\left(1+\frac{0,084}{12}\right)^{36} - 1\right]}{\frac{0,0084}{12}}$$

$$\frac{0,084}{12} \times 180\,000 = \frac{x\left[\left(1+\frac{0,084}{12}\right)^{36} - 1\right]}{\frac{0,0084}{12}} \times \frac{0,084}{12}$$

$$1260 = x\left(1+\frac{0,084}{12}\right)^{36} - 1$$

$$\frac{1260}{\left[\left(1+\frac{0,084}{12}\right)^{36} - 1\right]} = \frac{x\left[\left(1+\frac{0,084}{12}\right)^{36} - 1\right]}{\left[\left(1+\frac{0,084}{12}\right)^{36} - 1\right]}$$

$$x = 4413,82$$

Lerato will have to pay R4 413,82 per month

10. Ask learners if they have any questions before discussing the two situations in which the formula cannot be used as is.

11. Say: *Now that you have seen how the formula can be used as it is, in what is considered as the most common situation when paying an annuity, I am going to discuss when it needs to be adjusted slightly.*

Teaching note: The two explanations below assume that an annuity is started on the first of the month.

Situation 1:

If a person decides to make a payment immediately and then continues to pay one month later and continues regularly thereafter, the formula will be used as per normal, but an extra payment will have been made. Tell learners to write the summary below in their books and to add notes to help them remember why an extra payment is required:

Pay now (immediately) and then at the end of every month.
On a timeline: T_0
There will be one extra payment (n + 1)

Situation 2:

If a person decides to make a payment immediately and then at the beginning of the month from then onwards, the money will have to 'sit' in the bank for one month after the final payment is made (in the ordinary situation the final payment is made on the day the annuity ends). The number of payments will remain the same (for example, 12 payments in one year) but the total amount will 'grow' at the interest rate quoted for one more month.
Pay now and continue paying a month in advance. On a timeline: T_0 The final payment will 'sit' on the bank for another month and earn some interest. Treat it like the most common situation BUT 'grow' the answer using compound interest for another month.

12. Do two examples using these two situations with the learners.
    Learners should write the examples in their books, taking notes as they do so.

A young woman decides to start saving for her retirement. She immediately deposits R750 into an investment recommended by a friend. Thereafter, she continues to pay R750 at the end of each month for the next 15 years. The investment offers interest at 9,5% p.a. compounded monthly. Find the future value of her investment.

Note that the woman pays immediately and then continues to pay at the end of the month, which is the norm. There will be an extra payment.

Solution:

x = R750 ; $i = \frac{0,095}{12}$; n = (15 × 12) + 1 = 181

$$F_v = \frac{x[(1+i)^n - 1]}{i}$$

$$F_v = \frac{750\left[\left(1+\frac{0,095}{12}\right)^{181} - 1\right]}{\frac{0,095}{12}}$$

$$F_v = 300\ 062,42$$

The value of her investment will be R300 062,42

The owner of a transport company has just bought a new vehicle which he knows will need to be replaced in 4 years' time. He starts a sinking fund by immediately paying R4 500 into the investment which offers 10,5% p.a. compounded monthly. He continues to pay in advance which means the last payment will be paid one month before the annuity ends. He is expecting the new vehicle to cost about R300 000.

Will he have enough money in the sinking fund?

Note that the first payment is made immediately then the man continues to pay at the beginning of the month. His money will remain in the bank for one more month after the final payment and will therefore gain more interest.

Remind learners that the value of the annuity will become the principal amount that needs to gain more interest.

Compound interest formula:

$$A = P(1 + i)^n$$

The formula that will be used is:

$$F_v = \frac{x[(1+i)^n - 1]}{i}.(1+i)^n$$

Note that the principal amount to be grown for one more month is the future value.

Solution:

x = R4500 ; $i = \frac{0,105}{12}$; $n = (4 \times 12) = 48$

$$F_v = \frac{x[(1+i)^n - 1]}{i}$$

$$F_v = \frac{4500\left(1 + \frac{0,105}{12}\right)^{48} - 1}{\frac{0,105}{12}}.\left(1 + \frac{0,105}{12}\right)^1$$

$$F_v = 269\ 345{,}08$$

He will not have enough money in his sinking fund to replace the vehicle.

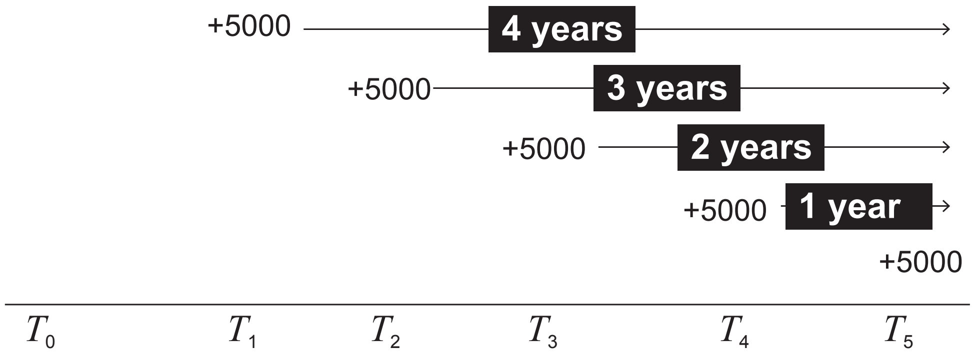
13. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

14. Give learners an exercise to complete on their own.

15. Walk around the classroom as learners do the exercise. Support learners where necessary.

16. Once learners have completed an exercise and it has been corrected spend some time showing them why the formula for future value works as it does.

17. Draw the following timeline representing a savings of R5 000 per year with the first payment being at the end of the year (as is the most common) at an interest rate of 5% p.a. compounded annually.



18. Tell learners to note that the first deposit of R5 000 will remain in the bank the longest and will therefore earn the most interest.

    Say: *Consider each amount on its own with the interest and add all the amounts together to find the total:*

$$5000(1+0{,}05)^4 + 5000(1+0{,}05)^3 + 5000(1+0{,}05)^2 + 5000(1+0{,}05)^1 + 5000$$

Note that the last R5 000 will not earn any interest – it will be deposited on the day that the annuity ends.

Ask learners if they agree that 1 + 2 + 3 = 3 + 2 + 1

If yes, ask learners to rearrange the above addition sequence and simplify.

$$\begin{aligned}&5000 + 5000(1+0{,}05)^1 + 5000(1+0{,}05)^2 + 5000(1+0{,}05)^3 + 5000(1+0{,}05)^4\\ &= 5000 + 5000(1{,}05) + 5000(1{,}05)^2 + 5000(1{,}05)^3 + 5000(1{,}05)^4\end{aligned}$$

Ask: *What do you recognise about this sequence?*

(It is a geometric series).

Ask: *What is the formula for the sum of a geometric series?*

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$a = 5000 \qquad r = \frac{T_2}{T_1} = 1{,}05 \qquad n = 5$$

$$S_n = \frac{5000((1{,}05)^5 - 1)}{1{,}05 - 1}$$

$$S_n = 27628{,}16$$

Ask learners to find the future value of the investment using the future value formula.

x = R5000 ; $i = 0{,}05$; $n = 5$

$$F_v = \frac{x[(1+i)^n - 1]}{i}$$

$$F_v = \frac{5000[(1+0{,}05)^5 - 1]}{0{,}05}$$

$$F_v = 27628{,}16$$

The value of her investment will be R27 628,16

Ask learners if they can see that the two formulae are the same.

$S_n = \frac{a(r^n - 1)}{r - 1}$ $S_n = \frac{5000((1{,}05)^5 - 1)}{1{,}05 - 1}$	$F_v = \frac{x[(1+i)^n - 1]}{i}$ $F_v = \frac{5000[(1+0{,}05)^5 - 1]}{0{,}05}$

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

http://learn.mindset.co.za/resources/mathematics/grade-12/finance-growth-and-decay/01-introducing-future-value-annuities

https://www.youtube.com/watch?v=joBu9TnFngQ

(future value with payment at the beginning of the annuity and a in advance thereafter)

https://www.youtube.com/watch?v=i5AwheXRA9Y

(deriving the future value formula)

TERM 1, TOPIC 3, LESSON 3

PRESENT VALUE ANNUITIES

Suggested lesson duration: 2,5 hours

POLICY AND OUTCOMES

A

CAPS Page Number	42
Lesson Objectives By the end of the lesson, learners should be able to: ● calculate the present value of annuity given monthly payment, interest and time period ● calculate the monthly payment using the present-value annuity formula.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Have Resource 8 ready for use in this lesson.
4. Write the lesson heading on the board before learners arrive.
5. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		CLASSROOM MATHS		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
4	88	2	66	2	85	4.4	97	4.4	102	3.4	124
5	92	4	70	5	91	4.7	105	4.5	108		
7	96	5	73	8	97			4.9	118		
		6	74					4.10	119		

C CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Once learners have learned about future value annuities, they should find present value annuities easier to understand.
2. Annuities (and, unfortunately, loans in particular) are part of everyday life for most people. Tell learners how understanding annuities will assist them once they start earning their own money and want to save.

DIRECT INSTRUCTION

1. Tell learners that in this lesson we will continue to learn about annuities but this time we will be learning about present value annuities which are loans.
2. Say: *it is common to need to work with loans in our lives. Most people cannot afford to pay cash for a house or a car and will need take a loan from a bank or a financial institution.*
3. Write the present value annuity formula on the board. Label each variable and discuss the variables with learners as you do so. Tell learners to write it in their books.

Regular payment

number of payments

$$P_v = \frac{x\,[1 - (1 + i)^{-n}]}{i}$$

Present value

interest rate

4. Tell learners to note the negative exponent. This can be used to assist them in remembering which formula is used for loans – the negative should remind them that it is a debt situation rather than a savings situation.
5. Say: *I am going to do two examples with you. Write them in your books and make notes as you do so.*

Example:
A teacher applies for a home loan. The bank charges 11,5% p.a. compounded monthly.
The teacher can afford to pay R6 000 per month. The bank offers a loan over 20 years.
The first payment will be made one month after the loan is granted.
Calculate the amount that the teacher can afford to borrow to the nearest rand.

Tell learners they should always write the formula down then state all the known variables.

Solution

$$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$

x = R6000 ; $i = \frac{0,115}{12}$; $n = 20 \times 12 = 240$

$$P = \frac{x[1-(1+i)^{-n}]}{i}$$

$$P = \frac{6000\left[1-\left(1+\frac{0,115}{12}\right)^{-240}\right]}{\frac{0,115}{12}}$$

$$P = 562\ 625{,}03$$

The teacher can afford to borrow an amount of R562 625,03

Example

John buys a car and needs to take a loan for R115 000. The bank charges 15,5% p.a. compounded monthly and is told the loan period will be 4 years.

Calculate John's monthly payment.

Remind learners that they should not round off until the final answer.

They should keep values in their calculator as they proceed with the calculation.

Solution:

$$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$

$P_v = 115\ 000$ $\quad i = \frac{0,155}{12}$ $\quad n = 48$

$$115\ 000 = \frac{x\left[1-\left(1+\frac{0,155}{12}\right)^{-48}\right]}{\frac{0,155}{12}}$$

$$\frac{0,155}{12} \times 115\,000 = \frac{x\left[1-\left(1+\frac{0,155}{12}\right)^{-48}\right]}{\frac{0,155}{12}} \times \frac{0,155}{12}$$

$$1485{,}416667 = x\left[1 - \left(1+\frac{0,155}{12}\right)^{-48}\right]$$

$$\frac{1485,416667}{\left[1-\left(1+\frac{0,155}{12}\right)^{-48}\right]} = \frac{x\left[1-\left(1+\frac{0,155}{12}\right)^{-48}\right]}{\left[1-\left(1+\frac{0,155}{12}\right)^{-48}\right]}$$

$$x = 3229{,}76$$

John's payments will be R3 229,76 per month

6. Ask learners if they have any questions.

Give learners an exercise (or some questions depending on the textbook your school uses) to do on straightforward present value questions similar to the examples above before moving on to deferred annuities and balance outstanding.

7. Once you have corrected the examples learners have completed on their own,
   Say: *Some advertisements offer a car for purchase, but the buyer is told he/she only needs to make the first payment in a few months' time.*
   This is called a deferred payment – payments are not started when the loan is granted.

8. Discuss with learners what they think will happen if this is the case.
   Ask questions such as:
   Do you think the company selling the car are just being generous?
   Once the payments are started, do you think you will still owe the same amount as you did a few months previously?

9. Explain that when payments are not started immediately, the amount loaned (borrowed) will become larger as interest will have been added for the time that payments were not made.

10. Do an example with learners to demonstrate this.
    Learners should write the example in their books and make notes as they do so.

Example
On the 1st of February, a student takes a loan to fund his studies for a one-year diploma. He agrees to start paying the loan back with equal monthly instalments for 4 years on the 31st of January the following year. The loan amount is R38 000 and the interest will be charged at 13% p.a. compounded monthly. Calculate the monthly payments.

Tell learners to note that:
- this is a loan and therefore the present value formula will be used;
- the first payment is not going to be paid at the end of the first month as usual. This one is only going to be paid 12 months later – this means the loan will grow during those 12 months to a new amount owing. The new amount will be the Present value in the formula.
- Before we can find the values for the present value formula, a calculation is required to find the value of the loan when the repayments start

Solution

$$A = P(1 + i)^n$$

$$A = 38000\left(1 + \frac{0,13}{12}\right)^{12}$$

$$A = \text{R}43\ 245,23$$

This amount will now be used as usual in the present value formula to find the payments.

P = R43 245,23 $i = \frac{0,13}{12}$ $n = (4 \times 12) = 48$

$$P = \frac{x[1 - (1 + i)^{-n}]}{i}$$

$$43245,23 = \frac{x\left[1 - \left(1 + \frac{0,13}{12}\right)^{-48}\right]}{\frac{0,13}{12}}$$

$$43245,23 = \frac{x\left[1 - \left(1 + \frac{0,13}{12}\right)^{-48}\right]}{\frac{0,13}{12}}$$

$$x = 1160,16$$

The student will pay R1 160,16 per month.

Give learners an exercise or some questions to do on deferred payments before moving on to balance outstanding.

11. Once learners have done the questions themselves, and you have corrected them, say: *Sometimes a person may have been paying a loan off for a while and would like to find out what is currently owed. Perhaps he/she has been paid a bonus and would like to use it to pay off a loan and therefore save on the interest charged.*

12. Do an example with learners to explain this situation further and find the balance outstanding at any given time.

Example

A loan of R60 000 is paid off over a period of 5 years by equal monthly instalments at an interest rate of 9,5% p.a. compounded monthly.
Determine the balance outstanding after 3 years.

Tell learners that first we need to find the monthly repayment so that we will be able to establish how much of the loan has been paid back in the 3 years.

$$P_v = \frac{x[1 - (1 + i)^{-n}]}{i}$$

$$60000 = \frac{x\left[1 - \left(1 + \frac{0,095}{12}\right)^{-60}\right]}{\frac{0,095}{12}}$$

$$x = 1260,11$$

Tell learners that now we can use this to find what all of the payments have grown to (as the interest has essentially been earned by the bank or person who gave the loan). The future-value formula will be required as these payments have accumulated to a future value from the beginning of the 3 years.

$$F_v = \frac{x[(1+i)^n - 1]}{i}$$

$$F_v = \frac{1260,11\left[\left(1 + \frac{0,095}{12}\right)^{36} - 1\right]}{\frac{0,095}{12}}$$

$$F_v = 52\ 251,42$$

Tell learners that the next step will be to find what the loan would have grown to in the three years (imagine this being the amount that would have been owed at this stage if no payments had been made – like a deferred payment). The compound interest formula will be used to find this value

$$A = P(1 + i)^n$$

$$A = 60000\left(1 + \frac{0,095}{12}\right)^{36}$$

$$A = \text{R}79\ 696,24$$

The difference between these two amounts is the balance outstanding.

R79 696,24 – 52 251,42 = 27 444,82

Show learners that the following formula can be used to find the balance outstanding, but it was done in steps to explain why certain calculations were done:

$$P(1+i)^n - \frac{x[(1+i)^n - 1]}{i}$$

13. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

14. Give learners an exercise to complete on their own.

15. Walk around the classroom as learners do the exercise. Support learners where necessary.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=Vs3loeKFbAo

(present and future value)

https://www.youtube.com/watch?v=tv1qKi7BEP4

TERM 1, TOPIC 3, LESSON 4

FINDING THE TIME PERIOD

Suggested lesson duration: 1,5 hours

POLICY AND OUTCOMES

A

CAPS Page Number	42
Lesson Objectives By the end of the lesson, learners should be able to find the: ● time period of an annuity ● final instalment of a present value annuity.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson have the first two equations from point 2 ready.
5. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		CLASSROOM MATHS		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
1	64	7	77	3	87	4.6	103	4.7 4.8	113 115	3.1	108

C

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. This lesson requires learners to work with logarithms.

DIRECT INSTRUCTION

1. Start the lesson by reminding learners how to find the value of an exponent using logarithms.
2. Write the following two examples on the board and do them with learners if necessary.

$2^{x-1} = 12$ $x - 1 = \log_2 12$ $x = \log_2 12 + 1$ $x = 4,585$	$7.3^{2x+1} = 56$ $3^{2x+1} = 8$ $2x + 1 = \log_3 8$ $2x = \log_3 8 - 1$ $x = \frac{\log_3 8 - 1}{2}$ $x = 0,446$

3. Tell learners that this skill will be required if the number of payments is required in a calculation involving the present or future value formula because this is represented by an exponent.
4. Say: *I am going to do two examples with you – one with each type of annuity. Write the examples in your books and make notes as you do so.*

Example Gloria would like to save for a holiday. She makes payments of R550 each month at an interest rate of 7,8% p.a. compounded monthly. How long will it take her to save R10 000? (Answer in months).
Tell learners to write down the formula to be used and list the known values. The equation will need to be manipulated until an exponential equation is found. After this, logarithms need to be used to find the exponent n. Remind learners not to round any answer until the final step.

Solution

$$F_v = \frac{x[(1+i)^n - 1]}{i}$$

$F_v = 10000$ $\quad x = R550 \quad$ $i = \frac{0,078}{12}$

$$10000 = \frac{550\left[\left(1+\frac{0,078}{12}\right)^n - 1\right]}{\frac{0,078}{12}}$$

$$\frac{0,078}{12} \times 10000 = \frac{550\left[\left(1+\frac{0,078}{12}\right)^n - 1\right]}{\frac{0,078}{12}} \times \frac{0,078}{12}$$

$$65 = 550\left[\left(1+\frac{0,078}{12}\right)^n - 1\right]$$

$$\frac{65}{550} = \frac{550\left[\left(1+\frac{0,078}{12}\right)^n - 1\right]}{550}$$

$$\frac{13}{110} = \left(1+\frac{0,078}{12}\right)^n - 1$$

$$\frac{13}{110} + 1 = \left(1+\frac{0,078}{12}\right)^n$$

$$n = \log_{\left(1+\frac{0,078}{12}\right)}\left(\frac{13}{110}+1\right)$$

$$n = 17,24$$

It will take Gloria a little over 17 months to save R10 000

Example

Sizwe takes a loan to pay off a debt. He borrows R15 000 and makes monthly payments of R400 at an interest rate of 11,25%p.a. compounded monthly.

How long will it take for him to pay back the loan? (Answer in months).

Tell learners to write down the formula to be used and list the known values. The equation will need to be manipulated until an exponential equation is found. After this, logarithms need to be used to find the exponent n. Remind learners not to round any answer until the final step.

Solution

$$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$

$$P_v = 15000 \qquad x = \text{R}400\ ;\ i = \frac{0,1125}{12}$$

$$15000 = \frac{400\left[1-\left(1+\frac{0,1125}{12}\right)^{-n}\right]}{\frac{0,1125}{12}}$$

$$\frac{0,1125}{12}\times 15000 = \frac{400\left[1-\left(1+\frac{0,1125}{12}\right)^{-n}\right]}{\frac{0,1125}{12}}\times\frac{0,1125}{12}$$

$$140{,}625 = 400\left[1-\left(1+\frac{0,1125}{12}\right)^{-n}\right]$$

$$\frac{140,625}{400} = \frac{400\left[1-\left(1+\frac{0,1125}{12}\right)^{-n}\right]}{400}$$

$$\frac{140,625}{400} = 1-\left(1+\frac{0,1125}{12}\right)^{-n}$$

Tell learners that it would be easier at this stage to rearrange the equation so that the power on the right-hand side is now on the left-hand side (using inverse operations) to ensure that they work with a positive term on each side of the equation. This can help to avoid careless mistakes.

$$\left(1+\frac{0,1125}{12}\right)^{-n} = 1-\frac{140,625}{400}$$

$$-n = \log_{\left(1+\frac{0,1125}{12}\right)}\left(1-\frac{140,625}{400}\right)$$

$$-n = -46{,}42$$

$$n = 46{,}42$$

It will take Sizwe 46,42 months to pay off the loan

5. Ask if there are any questions.

6. Say: *Note that in each of these answers the time period was not a whole number.*

7. Tell learners that sometimes a person will pay more than the regular amount required when paying back a loan. This is sound financial behaviour. If a loan is granted over 10 years and you have had a regular annual increase at work, it makes sense to rather pay more into your loan and pay it off sooner. This can relate to a large saving on interest. If paying extra is the case, there is a good chance the final payment will probably not be the same amount that has been regularly paid.

8. The idea for this is a similar to finding the balance outstanding. Say: *I will do an example to demonstrate. Write it in your books and make notes as you do so.*

Example

A newly married couple buy a house and take a loan for R800 000. The interest rate is 9,2% p.a compounded monthly and the term is 20 years.

a) Find the monthly instalment

b) The couple decides than they can afford to pay R8000 per month. How long will they take to pay of their loan?

c) Find the final payment.

a) By now this should be becoming easier for learners. The present value formula needs to be used to solve for x.

b) The present value formula needs to be used and n needs to be found using logarithms.

c) Tell learners to note that the house will be paid off in 190 full payments but when the 190th payment has been made, there will still be an amount outstanding. The same concept is used here as in finding outstanding balance. The loan must be grown to the time of interest – 190 months and the calculation needs to be made as to what all the payments so far have accumulated to in the 190 months using the future value formula. The difference between these two will be the balance outstanding at 190 months.

Ask: *Why would this not be the final payment?*

This is only the balance outstanding at 190 months – they will only make their final payment one month later. This amount will still have to have another month's worth of interest added on using the compound interest formula.

Solution

a) $P_v = \dfrac{x[1-(1+i)^{-n}]}{i}$

$$P_v = 800\ 000 \qquad i = \frac{0.092}{12} \qquad n = 20 \times 12 = 240$$

$$800000 = \frac{x\left[1-\left(1+\frac{0,092}{12}\right)^{-240}\right]}{\frac{0,092}{12}}$$

$$\frac{0.092}{12} \times 800000 = \frac{x\left[1-\left(1+\frac{0,092}{12}\right)^{-240}\right]}{\frac{0,092}{12}} \times \frac{0,092}{12}$$

$$6133,333\ldots = x\left[1-\left(1+\frac{0,092}{12}\right)^{-240}\right]$$

$$\frac{6133,333...}{\left[1-\left(1+\frac{0,092}{12}\right)^{-240}\right]} = x\,\frac{1-\left[\left(1+\frac{0,092}{12}\right)^{-240}\right]}{1-\left[\left(1+\frac{0,092}{12}\right)^{-240}\right]}$$

$$x = 7301,03$$

The monthly instalment will be R7301,03

b) $P_v = \frac{x[1-(1+i)^{-n}]}{i}$

$$P_v = 800000 \quad x = 8000 \quad i = \frac{0.092}{12}$$

$$800000 = \frac{8000\left[\left(1 - 1 + \frac{0,092}{12}\right)^{-n}\right]}{\frac{0,092}{12}}$$

$$\frac{0.092}{12} \times 800000 = \frac{8000\left[1-\left(1+\frac{0,092}{12}\right)^{-n}\right]}{\frac{0,092}{12}} \times \frac{0,092}{12}$$

$$\frac{18400}{3} = 8000\left[1–\left(1+\frac{0,092}{12}\right)^{-n}\right]$$

$$\frac{18400}{3} \div 8000 = 1-\left(1+\frac{0,092}{12}\right)^{-n}$$

$$\frac{23}{30} = 1-\left(1+\frac{0,092}{12}\right)^{-n}$$

$$\left(1+\frac{0,092}{12}\right)^{-n} = 1-\frac{23}{30}$$

$$-n = \log_{\left(1+\frac{0,092}{12}\right)}\left(1-\frac{23}{30}\right)$$

$$-n = -190,5467912$$

$$n = 190,5467912$$

c) Balance outstanding = $P(1+i)^n - \frac{x[(1+i)^n - 1]}{i}$

$$800000\left(1+\frac{0,092}{12}\right)^{190} - \frac{8000\left[\left(1+\frac{0,092}{12}\right)^{190} - 1\right]}{\frac{0,092}{12}}$$

$$= 4348,560386$$

$$A = P(1+i)^n$$

$$A = 4348,560386\left(1+\frac{0,092}{12}\right)^{1}$$

$$A = 4381,90$$

The final payment will be R4381,90

Have a discussion with learners about this situation. Point out how many months (and therefore year) of money were saved just by making the payment a bit higher than necessary.

This loan should have gone on for 49 months longer than it did which is more than 4 years!

9. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

10. Give learners an exercise to complete on their own.

11. Walk around the classroom as learners do the exercise. Support learners where necessary.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=xWrAg5A7n4k

https://www.youtube.com/watch?v=-qb_aKPscIQ

TERM 1, TOPIC 3, LESSON 5

DISCUSSION ON ANALYSING DIFFERENT LOAN OR SAVINGS OPTIONS AND PYRAMID SCHEMES

Suggested lesson duration: 0,5 hour

A POLICY AND OUTCOMES

CAPS Page Number	42

Lesson Objectives

By the end of the lesson, learners should be able to:

- compare loans or investments

and

- have a better understanding of pyramid schemes.

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**CLASSROOM MATHS**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
6	94	8 Assign	81 99	4	89	Assign	106	4.11	121	3.5	130

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Each textbook approaches this topic in a different way.
2. Prepare a lesson according to the textbook you use. It is important that you compare different possibilities for a savings or a loan.
3. Pyramid schemes need discussing for the sake of the learners gaining an understanding of the flaws of such schemes.
4. This lesson will take the form of an example of comparing options available for saving.
5. There are links to both articles and videos at the end of the lesson plan if you feel you need a better understanding of pyramid schemes before discussing them with learners.

DIRECT INSTRUCTION

1. Say: *It is important to use your knowledge of financial matters to make sound decisions regarding savings or loans. Sometimes you might need to consider two options carefully before deciding which one to choose.*
2. Do two fully worked examples concerning which option to choose from two savings plans. Say: *Write the examples in your books and make notes as you do so.*

Example 1

Determine which of the following investments is better:

Investment A: R1500 per month invested at an interest rate of 6,2% p.a. compounded monthly for 5 years

Investment B: R5000 per quarter invested at an interest rate of 6% p.a compounded quarterly for 4 years.

By now learners should find the calculations for each of these investments easy. Remind learners to write down the formula to be used (in this case the future value formula as it is a savings) and list the values given.

Once the calculations have been done, discuss the various options. Listen to learners' ideas and answers. Make it clear that the decision may not always be based on the sum of money at the end but rather on what it is needed for and when it may be needed.

Solution

$$F_v = \frac{x[(1+i)^n - 1]}{i}$$

Investment A: $x = 1500$ $\quad i = \frac{0,062}{12}$ $\quad n = 12 \times 5 = 60$

$$F_v = \frac{1500\left[\left(1 + \frac{0,062}{12}\right)^{60} - 1\right]}{\frac{0,062}{12}}$$

$$F_v = 105194,73$$

Investment A will be worth R105194,73 at the end of the 5 years

Investment B: $x = 6000$ $\quad i = \frac{0,06}{4}$ $\quad n = 4 \times 4 = 16$

$$F_v = \frac{6000\left[\left(1 + \frac{0,06}{4}\right)^{16} - 1\right]}{\frac{0,06}{4}}$$

$$F_v = 107594,22$$

Investment B will be worth R107 594,22 at the end of the 4 years

Note that the amounts are similar but one of them grew to that in 4 years.

Note too that the payment for investment B is considerably larger than the payment for investment A – R1 500 per month would be the same as R4 500 per quarter.

Example 2

You decide to take a bank loan to buy a car. The car costs R110 000. The following options are presented to you:

Option 1: An 8-year loan at 11,5% p.a. compounded monthly

Option 2: A 5-year loan at 10,5% p.a. compounded monthly

Option 3: A 3-year loan at 9,5% p.a. compounded monthly.

By now learners should find the calculations for each of these investments easy. Remind learners to write down the formula to be used (in this case the future value formula as it is a savings) and list the values given.

Once the calculations have been done, discuss the various options. Listen to learners' ideas and answers:

- Make it clear that the decision may not always be based on the final amount paid, but possibly on what could be afforded monthly.
- Discuss what disadvantage an 8-year loan could have – the car may need replacing by then or at the very least be rather a lot older by the time it is finally paid.
- The smaller instalments would appeal to most people but how much would each one of these options end up costing the buyer?

Solution

$$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$

Option 1:

$$P_v = 110000 \qquad i = \frac{0,115}{12} \qquad n = 12 \times 8 = 96$$

$$110000 = \frac{x\left[1-\left(1+\frac{0,115}{12}\right)^{-74}\right]}{\frac{0,115}{12}}$$

$$1054,166667 = x\left[1-\left(1+\frac{0,155}{12}\right)^{-96}\right]$$

$$x = \frac{1054,166667}{\left[1-\left(1+\frac{0,115}{12}\right)^{-96}\right]}$$

$$x = 1757,73$$

Option 2:

$$P_v = 110000 \qquad i = \frac{0,105}{12} \qquad n = 12 \times 5 = 60$$

$$110000 = \frac{x\left[1-\left(1+\frac{0,105}{12}\right)^{-60}\right]}{\frac{0,105}{12}}$$

$$962,5 = x\left[1-\left(1+\frac{0,105}{12}\right)^{-60}\right]$$

$$x = \frac{962,5}{\left[1-\left(1+\frac{0,105}{12}\right)^{-60}\right]}$$

$$x = 2364,33$$

Option 3:

$$P_v = 110000 \qquad i = \frac{0,095}{12} \qquad n = 12 \times 3 = 36$$

$$110000 = \frac{x\left[1-\left(1+\frac{0,095}{12}\right)^{-36}\right]}{i\,\frac{0,095}{12}}$$

$$\frac{5225}{6} = x\left[1-\left(1+\frac{0,095}{12}\right)^{-36}\right]$$

$$x = \frac{\frac{5225}{6}}{\left[1-\left(1+\frac{0,095}{12}\right)^{-36}\right]}$$

$$x = 3523,62$$

3. Ask learners if anyone has any questions.

4. Say: *You may have heard of schemes that are supposed to make you rich if you join. These are not usually viable options. Unless you are one of the first people to join, there is a strong chance you would lose, rather than make, money.*

5. Tell learners a little about pyramid schemes.

> If your textbook doesn't have enough information on pyramid schemes, there is more information in the links below.

6. Ask directed questions so that you can ascertain learners' level of understanding.
   Ask learners if they have any questions.

7. Give learners an exercise to complete with a partner.

8. Walk around the classroom as learners do the exercise. Support learners where necessary.

9. Once the exercise has been corrected, hold a short discussion on pyramid schemes.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

Articles:

https://money.howstuffworks.com/pyramid-scheme1.htm

https://www.huffingtonpost.co.za/2018/03/09/were-still-falling-for-ponzi-and-pyramid-schemes_a_23381530/

https://www.scamwatch.gov.au/types-of-scams/jobs-employment/pyramid-schemes

Videos:

https://www.youtube.com/watch?v=VVUUbEw_Pm8

https://www.youtube.com/watch?v=1QkZcdCDJJg

https://www.youtube.com/watch?v=y9rJZX72oIw

TERM 1, TOPIC 3, LESSON 6

REVISION AND CONSOLIDATION

Suggested lesson duration: 1 hour

POLICY AND OUTCOMES

A

CAPS Page Number	42
Lesson Objectives By the end of the lesson, learners will have revised: ● the concepts covered in Grade 12 finance.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson have the information for the questions ready.
5. If you feel that the learners require more practice, either use the revision exercise at the end of another Grade 12 textbook or items from a Grade 12 test on Finance.
6. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**CLASSROOM MATHS**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
Rev S Ch	100 102	Rev	82	Qu's	100	Rev	106	4.6 4.12 4.13	111 123 125	3.6	134

C CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Ask learners to recap what they have learned in this section. Point out issues that you know are important as well as problems that you encountered from your own learners.
2. If learners want you to explain a concept again, do that now.

DIRECT INSTRUCTION

This lesson is made up of fully worked examples from a past examination covering most of the concepts in this topic. As you work through these with the learners, it is important to frequently talk about as many concepts as possible.
For example, use the words future value, present value, monthly payment and interest rates.

Say: *I am going to do an entire Finance question from the 2016 final examination with you. You should write them down as I do them, taking notes at the same time.*

Questions:
On 1 June 2016 a bank granted Thabiso a loan of R250 000 at an interest rate of 15% p.a. compounded monthly, to buy a car. Thabiso agreed to repay the loan in monthly instalments commencing on 1 July 2016 and ending 4 years later on 1 June 2020. However, Thabiso was unable to make the first two instalments and only commenced with the monthly instalments on 1 September 2016.

a) Calculate the amount Thabiso owed the bank on 1 August 2016, a month before he paid his first monthly instalment.
b) Having paid the first monthly instalment on 1 September 2016, Thabiso will still pay his last monthly instalment on 1 June 2020. Calculate his monthly instalment.
c) If Thabiso paid R9 000 as his monthly instalment starting 1 September 2016, how many months sooner will he repay his loan?
d) If Thabiso paid R9 000 as his monthly instalment starting 1 September 2016, calculate the final instalment to repay the loan.

Teaching notes

a) Remind learners that the situation described here is referred to as a deferred payment.
Ask: *What do we need to do to find the new amount owing?*
(Use the compound interest formula to grow the money for the 2 months missed).

b) Ask: *Which formula will be required?*
(Present-value – it is a loan)
Ask: *Do you have the present value, the interest and the number of payments?*
(Yes – will need to calculate the number of payments first though).

c) Point out that R9000 is larger than the amount found in (b) and hence is the reason he could pay it off earlier.

d) Ask: *Which formula will be required?*
(Present-value – it is a loan).
As we are finding n (the number of payments):
Ask: *do you have the present value, the monthly payment and the interest?*
(Yes). Note that logs will be required to solve for the exponent.
Once the answer has been found, spend a few minutes discussing this with learners.
35,41 means that there will be 35 full payments of R9 000, but it won't be fully paid yet.
Hence, learners need to say it will take 36 months to pay as there will still be one payment in the 36th month to finalise to loan being paid in full.

e) This question is linked to the outstanding balance. We need to find the amount owing after the 35th payment. Remind learners of the steps to follow:
- Grow the loan to the time concerned (35 months).
- Use the future value formula to calculate what all the payments have grown to so far. Subtract this from the loan grown amount. This is the outstanding balance. However, point out that he isn't going to pay this amount right now (in the 35th month), he will only pay it in a month's time so the compound interest formula will be required to find the full amount owing.

Solutions:

a)
$$A = P(1 + i)^n$$
$$A = 250000\left(1 + \frac{0,15}{12}\right)^2$$
$$A = \text{R}256\ 289,06$$

b) 1 September 2016 to 1 June 2020 – 46 payments

$$P = x\left[\frac{(1 - (1 + i)^{-n}}{i}\right]$$
$$256289,06 = x\left[\frac{1 - \left(1 + \frac{0,15}{12}\right)^{-46}}{\frac{0,15}{12}}\right]$$
$$x = \text{R}7359,79$$

c) $$P = x\left[\frac{1-(1+i)-n}{i}\right]$$

$$256289{,}06 = 9000\left[\frac{1-\left(1+\frac{0{,}15}{12}\right)^{-n}}{\frac{0{,}15}{12}}\right]$$

$$\left(1+\frac{0{,}15}{12}\right)^{-n} = 0{,}6440$$

$$-n = \log_{\left(1+\frac{0{,}15}{12}\right)} 0{,}6440$$

$n = 35{,}41$ $\therefore$ 36 payments (10 months sooner)

d) Outstanding balance:

$$256289{,}06\left(1+\frac{0{,}15}{12}\right)^{35} - 9000\left[\frac{\left(1+\frac{0{,}15}{12}\right)^{35} - 1}{\frac{0{,}15}{12}}\right]$$

= R3735,45

Final payment in 36th month: $3735{,}45\left(1+\frac{0{,}15}{12}\right)^{1}$

= R3782,14

1. Ask directed so that you can ascertain learners' level of understanding. Ask learners if they have any questions.
2. Give learners an exercise to complete on their own.
3. Walk around the classroom as learners do the exercise. Support learners where necessary.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=Ze0EnCbS_3s

https://www.youtube.com/watch?v=iuBpWTJbi1E

Term 1, Topic 4: Topic Overview

TRIGONOMETRY

A. TOPIC OVERVIEW

- This topic is the fifth of five topics in Term 1.
- This topic runs for two weeks (9 hours).
- It is presented over seven lessons.
- The lessons have been divided according to sub-topics, not according to one school lesson. An approximate time has been allocated to each lesson (which will total 9 hours). For example, one lesson in this topic could take two school lessons. Plan according to your school's timetable.
- Trigonometry counts 33% of the final Paper 2 examination.
- This section covers compound and double angles as well as identities, reductions and general solutions.
- Algebraic manipulation such as substitution and solving equations are important skills for this section.

Breakdown of topic into 7 lessons:

	Lesson title	**Suggested time (hours)**		**Lesson title**	**Suggested time (hours)**
1	Revision	2	5	General Solutions	1
2	Investigation – compound and double angles	1	6	Identities	1
3	Compound and Double angles	2	7	Revision and Consolidation	1
4	Derivation and application of compound angles and double angles	1			

B SEQUENTIAL TABLE

GRADE 11 and earlier	GRADE 12
LOOKING BACK	**CURRENT**
● Definition of the trigonometric ratios (sinθ,cosθ,tanθ) in right-angled triangles ● Extend the definitions of the trig ratios to $0^\circ \leq \theta \leq 360^\circ$ ● Use the special angles without the use of a calculator ● Define the reciprocals of the trig ratios ● Solve problems in 2 dimensions ● Derive and use the identities: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\sin^2\theta + \cos^2\theta = 1$ ● Derive the reduction formulae ● Determine general solutions and specific solutions of trig equations ● Establish sine, cosine and area rules and use them to solve problems in 2 dimensions	● Proof and use of compound angles identities ● Proof and use of double angle identities ● Solve problems in 3 dimensions (covered in the next topic)

C WHAT THE NSC DIAGNOSTIC REPORTS TELL US

According to **NSC Diagnostic Reports** there are a number of issues pertaining to Trigonometry.

These include:

- Incorrect selection and application of reduction formulae
- Ignoring the instruction ‘without the use of a calculator’
- Algebraic skills (such as using brackets in substitution) let learners down
- Not recognising compound angles or double angles.

It is important that you keep these issues in mind when teaching this section.

While teaching Trigonometry, it is important to remind learners that, in this section, knowledge of theory is integral in answering questions. Encourage learners to show all steps in a calculation.

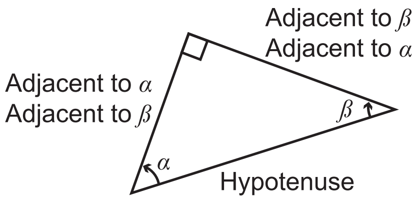
ASSESSMENT OF THE TOPIC

- CAPS formal assessment requirements for Term 1:
    - Investigation/Project
    - Assignment
    - Test
- Two tests, each with a memorandum, are provided in the Resource Pack. The tests are aligned to CAPS in every respect, including the four cognitive levels as required by CAPS (page 53).
- The questions usually take the form of using reductions, compound angles and double angles to simplify expressions; Trigonometric functions; finding general solutions; and proving identities.
- Monitor each learner's progress to assess (informally) their grasp of the concepts. This information can form the basis of feedback to the learners and will provide you valuable information regarding support and interventions required.

MATHEMATICAL VOCABULARY

Be sure to teach the following vocabulary at the appropriate place in the topic:

Term	**Explanation**
Trigonometry	Trigonometry is concerned with the measurement and calculation of the lengths of sides and the sizes of angles of triangles Greek letters are often used to stand for angles in triangles
right-angled triangle	Triangle with one angle of 90º (a right angle)
hypotenuse	The longest side in a right-angled triangle It is always opposite the right angle
opposite	The side opposite any angle you are currently dealing with in a triangle
adjacent	The side next to any angle you are currently dealing with in a triangle



quadrants	The four areas made when a Cartesian plane is drawn with an x-axis and a y-axis
reduction	Re-writing an expression into a simpler form using acute angles
identity	An equation that is true for all values of the variables
invalid identity	An identity that cannot be true if certain values are used in the variable's place For example, if the value made a denominator = 0.
general solution	The formula which represents all possible solutions to a trigonometric equation by taking the period of the function into account
compound angle	A compound angle 'formula' is a trigonometric identity which expresses a trigonometric function of $(A + B)$ or $(A - B)$ in terms of trigonometric functions of A and B
double angle	Double-angle formulae allow the expression of trigonometric functions of angles equal to 2α to be written in terms of α, which can simplify the functions and make it easier to perform more complex calculations

TERM 1, TOPIC 4, LESSON 1

REVISION

Suggested lesson duration: 2 hours

POLICY AND OUTCOMES

A

CAPS Page Number	42
Lesson Objectives By the end of the lesson, learners will have revised: ● Grade 11 Trigonometry.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Have Resource 9 ready for use during this lesson.
4. Write the lesson heading on the board before learners arrive.
5. If there isn't a revision exercise in the textbook that you use, or the exercise in your textbook is not detailed enough, either use the revision exercise at the end of a Grade 11 textbook or items from a Grade 11 test on Trigonometry.
6. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**CLASSROOM MATHS**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
Rev 10 11	110 133 136			Qu's 6	105 130	6.1 – 6.3	125 - 127	5.1 – 5.4	129 - 136	4.1	142

C

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. If learners are not confident in their Grade 11 work, they will find the Grade 12 work extremely difficult. For this reason this revision lesson is of the utmost importance. Ensure that each concept is covered well and that learners use the opportunity to ask questions and confirm that they understand all previous work.
2. Each concept from Grade 11 is repeated in Grade 12 – but will now include double angles and compound angles. For example, proving an identity will still be assessed but it may include knowing how to expand a double angle.

DIRECT INSTRUCTION

This lesson will take the form of working through a set of notes with learners. If photocopying is possible, this would be ideal. Learners can then work on the notes and have them available for reference throughout the topic.
The notes are available in the Resource Pack – Resource 9.

1. Ask learners what they remember learning in trigonometry in Grade 11. Write topics on the board as learners give them. The following topics should be on the list:
    - Reductions
    - Special angles
    - Angles in other quadrants/Pythagoras questions
    - General solution
    - Identities
    - Invalid identities
2. Say: *We are going to revise each of these concepts. Make sure you write down any examples we do and make notes as you do so. These concepts are essential to understanding Grade 12 Trigonometry.*
3. Hand the notes out now (if photocopying was possible) or start working through the notes on the chalkboard.
4. Ask directed questions throughout the lesson while working through the notes so that you can ascertain learners' level of understanding. Once all notes and examples are complete, ask learners if they have any questions.
5. Give learners an exercise to complete with a partner.

6. Walk around the classroom as learners do the exercise. Support learners where necessary.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=PgubJofzbvs

https://www.youtube.com/watch?v=bRSrA3Wf8FI

TERM 1, TOPIC 4, LESSON 2

INVESTIGATION

Suggested lesson duration: 1 hour

A POLICY AND OUTCOMES

CAPS Page Number	42
Lesson Objectives By the end of the lesson, learners will have: ● completed an investigation on compound angles and double angles.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the investigation.
3. Make copies of the investigation for each learner. The investigation is available in the Resource Pack – Resource 10.
4. It is important that you mark this investigation immediately. Preferably, it should be returned by the following lesson but by Lesson 4 at the latest. It will be useful for learners to have their own investigation with them when the formulae are derived.
5. If it is not possible for you to mark it yourself, mark it with learners.

C CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. An investigation is an activity that should lead a learner to a deeper understanding of a mathematical concept.
2. This investigation deals with compound angles and double angles.
3. This is not meant to be an investigation for marks. It is an exercise for learners to realise why the rules that they are about to learn really work.

DIRECT INSTRUCTION

1. Hand out the investigation to each learner.
2. Tell learners that they must work on their own and that they will have 45 minutes to complete the investigation.

MARKING RUBRIC

Investigation – Trigonometry

PART 1

Complete the table. Use a scientific calculator. Write your answer to THREE decimal places. Answers only are acceptable.

Let A = 20º and B = 10º	
cos (A + B)	0,866
cos A + cos B	1,925
cos (A – B)	0,985
cos A – cos B	–0,045
cos A.cos B + sin A.sin B	0,985
cos A.cos B – sin A.sin B	0,866
Let A = 15º and B = 35º	
sin (A + B)	0,766
sin A + sin B	0,832
sin (A – B)	–0,342
sin A – sin B	–0,315
sin A.cos B + cos A.sin B	0,766
sin A.cos B – cos A.sin B	-0,342

Look carefully at your answers on the right-hand side and make conclusions about the expressions on the left-hand side.
Complete the statements by filling in = or ≠ in the space provided.

$\cos(A + B)$	$\boxed{\neq}$	$\cos A + \cos B$	$\sin(A + B)$	$\boxed{\neq}$	$\sin A + \sin B$
$\cos(A - B)$	$\boxed{\neq}$	$\cos A - \cos B$	$\sin(A - B)$	$\boxed{\neq}$	$\sin A - \sin B$
$\cos(A + B)$	$\boxed{=}$	$\cos A.\cos B - \sin A.\sin B$	$\sin(A + B)$	$\boxed{=}$	$\sin A.\cos B - \cos A.\sin B$
$\cos(A - B)$	$\boxed{=}$	$\cos A.\cos B + \sin A.\sin B$	$\sin(A - B)$	$\boxed{=}$	$\sin A.\cos B + \cos A.\sin B$

For those statements that you have concluded are equal, choose two different values for A and B to confirm that you are correct.

Statement that seems to be true:	$\cos(A + B) = \cos A.\cos B + \sin A.\sin B$
New values tested: A = B =	LHS = RHS =
Conclusion:	The statement above **is true**/false
Statement that seems to be true:	$\cos(A - B) = \cos A.\cos B + \sin A.\sin B$
New values tested: A = B =	LHS = RHS =
Conclusion:	The statement above **is true**/false
Statement that seems to be true:	$\sin(A + B) = \sin A.\cos B + \cos A.\sin B$
New values tested: A = B =	LHS = RHS =
Conclusion:	The statement above **is true**/false
Statement that seems to be true:	$\sin(A - B) = \sin A.\cos B - \cos A.\sin B$
New values tested: A = B =	LHS = RHS =
Conclusion:	The statement above **is true**/false

Using a scientific calculator, complete the following table. Write your answer to THREE decimal places. Answers only are acceptable.

Let $A = 20^\circ$	
$\sin 2A$	0,643
$2\sin A.\cos A$	0,643
Let $A = 50^\circ$	
$\sin 2A$	0,985

$2 \sin A . \cos A$	0,985

Before completing the next table, remember that $\cos^2 10^\circ = (\cos 10^\circ)^2$.
Be careful when using your calculator.

Let $A = 20^\circ$	
$\cos 2A$	0,766
$2 \cos A - 1$	0,766
$1 - 2\sin^2 A$	0,766
$\cos^2 A - \sin^2 A$	0,766
Let $A = 50^\circ$	
$\cos 2A$	$-0,174$
$2 \cos A - 1$	$-0,174$
$1 - 2\sin^2 A$	$-0,174$
$\cos^2 A - \sin^2 A$	$-0,174$

The table above should show that:

$\sin 2A = 2 \sin A . \cos A$	$\cos 2A = 2\cos^2 A - 1$ $\cos 2A = 1 - 2\sin^2 A$ $\cos^2 A - \sin^2 A$

Complete the following:

$\sin^2 A + \cos^2 A = 1$
$\cos(90^\circ - A) = \sin A$
$\sin(90^\circ - A) = \cos A$
$\cos(90^\circ + A) = -\sin A$
$\sin(90^\circ + A) = \cos A$

These will be used in a later lesson to derive all the identities that you have proved to be true.

TERM 1, TOPIC 4, LESSON 3

COMPOUND ANGLES AND DOUBLE ANGLES

Suggested lesson duration: 2 hours

A POLICY AND OUTCOMES

CAPS Page Number	42

Lesson Objectives

By the end of the lesson, learners should be able to:

- recognise compound angles and double angles
- expand and simplify compound angles and double angles.

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson have the summary of all the compound and double angle identities ready.
5. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

Compound angles

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**CLASSROOM MATHS**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
1 2	113 115	1	87	1 2 (1, 2, 5, 6)	111 114	5.1 5.2	111 112	5.5 5.6	138 139	4.2	149

Double angles

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		CLASSROOM MATHS		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
3 4	117 119	2	90	3	119	5.5	118	5.7	142	4.3	154

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Learners investigated the compound angle identities and tested the double angle identities in the previous lesson.
2. This lesson will consolidate those concepts and allow learners to practice them.

DIRECT INSTRUCTION

1. Start the lesson by writing the compound and double angle identities on the board.

<table>
<tr><td colspan="2">Compound angle identities</td></tr>
<tr><td>$\cos(A + B) = \cos A.\cos B - \sin A.\sin B$
$\cos(A - B) = \cos A.\cos B + \sin A.\sin B$</td><td>$\sin(A + B) = \sin A.\cos B + \cos A.\sin B$
$\sin(A - B) = \sin A.\cos B - \cos A.\sin B$</td></tr>
<tr><td colspan="2">Double angle identities</td></tr>
<tr><td>$\sin 2A = 2\sin A.\cos A$</td><td>$\cos 2A = \cos^2 A - \sin^2 A$
$\cos 2A = 2\cos^2 A - 1$
$\cos 2A = 1 - 2\sin^2 A$</td></tr>
</table>

2. Say: *Although these identities were tested using values in the investigation, we have not yet derived them to show where they come from and hence understand them better. We will do this in the next lesson. In this lesson, we are going to accept that they are true and work with them to become more familiar with them.*
3. Tell learners to write the summary in their books.
4. Say: *We are going to work through several examples now to ensure you feel confident in the use of these new rules. The first set of examples involves compound angles.*

TOPIC 4, LESSON 3: COMPOUND ANGLES AND DOUBLE ANGLES

Learners should write each example in their books making notes as they do so.
Note: ALL these examples must be done WITHOUT THE USE OF A CALCULATOR.

Example:	Teaching notes:
Simplify: sin 5º.cos 40º + cos 5º.sin 40º = sin (5º + 40º) = sin 45º $= \frac{\sqrt{2}}{2}$	Tell learners to look at their summary to find the identity that will assist them in simplifying. sin $(A + B)$ = sin A.cos B + cos A.sin B Once that has been done, a knowledge of special angles is required.
Simplify: cos $(B$ + 60º) + cos $(B$ – 60º) = cos B.cos 60º – sin B.sin 60º + cos B.cos 60º + sin B.sin 60º = 2 cos B.cos 60º = 2 cos $B.\left(\frac{1}{2}\right)$ = cos B	Point out that two of the identities will be required for this question. Once both expressions have been expanded, algebraic skills and special angles need to be used.
Prove that: cos (90º – A) = sin A using compound angles. cos (90º – A) = cos 90º.cos A + sin 90º.sin A = 0.cos A + 1.sin A = sin A	Point out that the aim is to use one of the identities to expand cos (90º – A). Special angles will be used again to show that the answer is sin A.
Use the compound angle identities to find the reduction formulae for: a) cos (180º – θ) b) sin (360º – β) a) cos (180º – θ) = cos 180º.cos θ + sin 180º.sin θ = (-1).cos θ + 0.sin θ = –cos θ b) sin (360º – β) = sin 360º.cos β – cos 360º.sin β = 0.cos β – 1.sin β = –sin β	These are very similar to the previous example. Encourage learners to first simplify by using their reduction formulae then do the question the longer way and confirm they got the same answer.

Simplify: sin 169º.sin 41º + sin 79º.sin 131º = sin (180º – 11º).sin 41º + sin 79º.sin(180º – 49º) = sin 11º.sin 41º + sin 79º.sin 49º = sin 11º.sin 41º + cos 11º.cos 41º = cos 11º.cos 41º + sin 11º.sin 41º = cos(11º – 41º) = cos(–30º) = cos 30º $= \frac{\sqrt{3}}{2}$	Tell learners that before becoming concerned that this does not look like any identity in their summary, they should first use reductions to ensure all angles are acute angles. Once this has been done, some thought is required. Remind learners of co-ratios. Check whether changing some of the angles to their co-ratios would help. It is a good idea to write the co-ratio of each angle in pencil above the current ratio then look at all the possibilities with a focus on both the pencil ratios and the original ratios.
Find the value of cos 15º. Leave your answer in simplest surd form. cos 15º = cos(45º – 30º) = cos 45º.cos 30º + sin 45º.sin 30º $= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$ $= \frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}$ $= \frac{\sqrt{6}+\sqrt{2}}{4}$	Tell learners that when they are asked to find the value of a trigonometric ratio without a calculator that is not a special angle there must be a way to use special angles to equal the degrees given. This should lead to a compound angle that can be simplified. Ask: *What special angle makes a sum or difference to equal* 15º*?* 45º – 30º

5. Ask if learners have any questions. Give learners an exercise to do on compound angles.
6. Once the exercise has been completed and corrected, work through another set of examples involving the double angle identities.
   Learners should write the examples in their books, making notes as they do so.
   Note: ALL these examples must be done WITHOUT THE USE OF A CALCULATOR.

Example:	Teaching notes:
Expand and simplify if possible using double angles: $\sin 120^{\circ}$ $= \sin 2(60^{\circ})$ $= 2 \sin 60^{\circ}.\cos 60^{\circ}$ $= 2.\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$ $= \frac{\sqrt{3}}{2}$	Although this could be done using reductions, the aim is to practice double angles. Ask: *How can we rewrite 120º?* (2(60º)). Say: *Expand the double angle and use special angles.*
Simplify:	
$1 - 2\sin^{2} 75^{\circ}$ $= \cos^{2}(75^{\circ})$ $= \cos 150^{\circ}$ $= \cos(180^{\circ} - 30^{\circ})$ $= -\cos 30^{\circ}$ $= -\frac{\sqrt{3}}{2}$	Learners need to recognise that this is a cosine double angle. Once this has been done, reductions can be used. (Learners may have wanted to expand $\cos(180^{\circ} - 30^{\circ})$ – this is acceptable at this stage but point out that unless they were specifically asked to do so, this takes much longer.
a) $\cos^{2} 30^{\circ} - \sin^{2} 30^{\circ}$ b) $\cos^{2} 30^{\circ} + \sin^{2} 30^{\circ}$ a) $\cos^{2} 30^{\circ} - \sin^{2} 30^{\circ}$ $= \cos 60^{\circ}$ $= 1/2$ b) $\cos^{2} 30^{\circ} + \sin^{2} 30^{\circ} = 1$	a) Learners need to recognise that this is a cosine double angle. b) Learners need to make sure they don't quickly assume that this is a cosine double angle now that they have learned these new identities. Remind learners to be aware that they still need to consider other identities.
$\frac{1 - 2\sin^{2}30^{0}}{2\cos^{2}15^{0} - 1}$ $= \frac{\cos 60^{0}}{\cos 30^{0}}$ $= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$ $= \frac{1}{2} \times \frac{2}{\sqrt{3}}$ $= \frac{1}{\sqrt{3}}$	Learners need to recognise that both the numerator and denominator are cosine double angles.

$\sin 22,5^\circ.\cos 22,5^\circ$ $= \frac{1}{2}.2\sin 22,5^\circ.\cos 22,5^\circ$ $= \frac{1}{2}\sin 45^\circ$ $= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$ $= \frac{\sqrt{2}}{4}$	This is a common problem to solve (and will appear within identities and other parts of trigonometry later on). Learners should notice that although this looks a little like the sine double angle it is not. Ask: *What is different?* (There isn't a 2 in front). Say: *To assist us in simplifying, we need to 'force' the double angle.* Ask: *What would be wrong with just putting a '2' in front?* (The expression will no longer be equal). Ask: *What can we do to 'reverse' putting a '2' in front to ensure that the expressions remain equal?* (Multiply by $\frac{1}{2}$).
sin 22,5º. Leave your answer in simplest surd form. If $\cos 2A = 1 - 2\sin^2 A$ then $\cos 45^\circ = 1 - 2\sin^2 22,5^\circ$ $\frac{\sqrt{2}}{2} = 1 - 2\sin^2 22,5^\circ$ $\frac{\sqrt{2}}{2} - 1 = -2\sin^2 22,5^\circ$ $\frac{\sqrt{2}-2}{2} = -2\sin^2 22,5^\circ$ $\frac{\sqrt{2}-2}{2} \div -2 = \frac{-2\sin^2 22,5^0}{-2}$ $\frac{\sqrt{2}-2}{2} \times -\frac{1}{2} = \sin^2 22,5^\circ$ $\frac{-(\sqrt{2}-2)}{4} = \sin^2 22,5^\circ$ $\sqrt{\frac{-(\sqrt{2}-2)}{4}} = \sqrt{\sin^2 22,5^0}$ $\frac{\sqrt{-\sqrt{2}+2}}{2} = \sin 22,5^\circ$	Note: This is a difficult question. We have used as many steps as possible to ensure learners can follow as closely as possible as you complete it on the board. Ask: *Consider the special angles that you know. What does the angle 22,5º remind you of?* (45º). Ask: *What statement could you make from the new identities that you know that has* 'sin 22,5º' *and 45º in the statement somewhere? The* 'sin 22,5º' *does not have to be by itself.* $\cos 45^\circ = 1 - 2\sin^2 22,5^\circ$ Once this has been established learners need to use algebraic skills and a knowledge of special angles to solve for sin 22,5º.

7. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

8. Give learners an exercise to complete on their own.
9. Walk around the classroom as learners do the exercise. Support learners where necessary.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=pj4RVYSRpMs

TERM 1, TOPIC 4, LESSON 4

DERIVATION AND APPLICATION OF COMPOUND ANGLES AND DOUBLE ANGLES FORMULAE

Suggested lesson duration: 1 hour

POLICY AND OUTCOMES

A

CAPS Page Number	42

Lesson Objectives

By the end of the lesson, learners will be able to:

- derive the compound angles and double angles formulae
- apply the compound angles and double angles formulae.

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson have the seven questions from point 4 ready.
5. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		CLASSROOM MATHS		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
5	123	3	92	2 (3, 4, 7-17)	114	5.3 5.6	114 120	5.8	144		

C CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. It is always important to show learners where a formula or identity comes from. This adds to their understanding of the topic.
2. In the case of the derivation of the compound and double angle identities, they are also examinable.

DIRECT INSTRUCTION

1. Start the lesson by writing the following on the board:

$$\cos(A - B) = \cos A.\cos B + \sin A.\sin B$$

2. Tell learners that by using (and accepting) this identity, the other identities that they have been learning about and using in the last few lessons can all be derived.

> To prove $\cos(A - B) = \cos A.\cos B + \sin A.\sin B$, the unit circle, distance formula and the cosine rule are used. This proof is not required.

3. Say: *There are some reductions that need to be used. We will go over these first.*
4. Write the following on the board and ask learners to simplify all of them.

$\sin^2 A + \cos^2 A =$	1
$\cos(90° – A) =$	$\sin A$
$\sin(90° – A) =$	$\cos A$
$\cos(90° + A) =$	$–\sin A$
$\sin(90° + A) =$	$\cos A$
$\sin(– A) =$	$–\sin A$
$\cos(–A) =$	$\cos A$

5. Do each of the following proofs with learners.
   Learners should write the proofs their books and make notes as they do so.

<table>
<tr><td colspan="2">Using $\cos (A - B) = \cos A.\cos B + \sin A.\sin B$ to prove:
$\cos (A + B) = \cos A.\cos B - \sin A.\sin B$</td></tr>
<tr><td>$\cos (A + B)$
$= \cos (A - (-B))$
$= \cos A.\cos (-B) + \sin A.\sin (-B)$
$= \cos A.\cos B + \sin A.(-\sin B)$
$= \cos A.\cos B - \sin A.\sin B$</td><td>$\cos (A + B)$ needs to be rewritten to look like $\cos (A - B)$
Once this has been done, follow the original rule to expand and simplify.
Reducing negative angles is required.</td></tr>
<tr><td colspan="2">Using $\cos (A - B)= \cos A.\cos B + \sin A.\sin B$ to prove:
$\sin (A + B) = \sin A.\cos B + \cos A.\sin B$</td></tr>
<tr><td>$\sin (A + B)$
$= \cos [90^\circ - (A + B)]$
$= \cos [90^\circ - A - B)]$
$= \cos [(90^\circ - A) - B)]$
$=\cos (90^\circ - \mathrm{A}).\cos B + \sin (90^\circ - A).\sin B$
$= \sin A.\cos B + \cos A.\sin B$</td><td>Use co-ratios to rewrite $\sin (A + B)$ to look like $\cos (A - B)$.
Once this has been done, use the distributive law to remove the brackets and regroup the terms.
Follow the $\cos (A + B)$ to expand and simplify.</td></tr>
<tr><td colspan="2">Using $\cos (A + B) = \cos A.\cos B - \sin A.\sin B$ to prove:
$\sin (A - B) = \sin A.\cos B - \cos A.\sin B$</td></tr>
<tr><td>$\sin (A - B)$
$= \cos [90^\circ - (A - B)]$
$= \cos [90^\circ - A + B)]$
$= \cos [(90^\circ - A) + B)]$
$= \cos (90^\circ - A).\cos B - \sin (90^\circ - \mathrm{A}).\sin B$
$= \sin A.\cos B - \cos A.\sin B$</td><td>Use co-ratios to rewrite $\sin (A - B)$ to look like $\cos (A + B)$. It may not look like $\cos (A + B)$ at first but once the distributive law has been used to remove brackets and the terms have been regrouped, it will.
Follow the $\cos (A + B)$ to expand and simplify.</td></tr>
</table>

6. Go through each proof once more. Ask learners if they have any questions before moving on to the double angle proofs.

Using $\sin(A + B) = \sin A.\cos B + \cos A.\sin B$ to prove: $\sin 2A = 2 \sin A.\cos A$	
$\sin 2A$ $= \sin(A + A)$ $= \sin A.\cos A + \cos A.\sin A$ $= 2 \sin A.\cos A$	$\sin 2A$ needs to be rewritten to look like $\sin(A + B)$ Once this has been done, follow the original rule to expand and simplify.
Using $\cos(A + B) = \cos A.\cos B - \sin A.\sin B$ to prove: $\cos 2A = \cos^2 A - \sin^2 A$	
$\cos 2A$ $= \cos(A + A)$ $= \cos A.\cos A - \sin A.\sin A$ $= \cos^2 A - \sin^2 A$	$\cos 2A$ needs to be rewritten to look like $\cos(A + B)$ Once this has been done, follow the original rule to expand and simplify.

7. Remind learners that there are two more versions of $\cos 2A$. Show how these are derived.

Remind learners of the square identity: $\sin^2 A + \cos^2 A = 1$ Ask learners to write this identity in terms of $\sin^2 A$ $\sin^2 A = 1 - \cos^2 A$ Ask learners to write this identity in terms of $\cos^2 A$ $\cos^2 A = 1 - \sin^2 A$ Tell learners that these will be used to find the other two versions of cos 2A	
$\cos 2A = \cos^2 A - \sin^2 A$ $= \cos^2 A - (1 - \cos^2 A)$ $= \cos^2 A - 1 + \cos^2 A$ $= 2\cos^2 A - 1$	Use $\sin^2 A = 1 - \cos^2 A$ to rewrite the right-hand side.
$\cos 2A = \cos^2 A - \sin^2 A$ $= 1 - \sin^2 A - \sin^2 A$ $= 1 - 2\sin^2 A$	Use $\cos^2 A = 1 - \sin^2 A$ to rewrite the right-hand side.

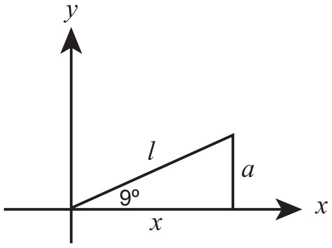
8. Point out that these identities are the only new concept that they will learn in trigonometry in Grade 12. All the sub-topics done in Grade 11, and the revision lesson at the beginning of this topic, will be asked again in Grade 12. The difference now is that they will have to deal with compound angles and double angles in identities, general solutions and Pythagoras type questions.

9. Identities and general solution questions will be covered in a lesson of their own.
   Tell learners that you will do two examples showing how double angles and compound angles can be used in Pythagoras type questions.
   Learners should write it in their books and take notes as they do so.

<table>
<tr><th>Example 1</th><th>Teaching notes</th></tr>
<tr><td>

If sin 9º = a, find sin 18º in terms of a
(without the use of a calculator)
Solution:

$\sin 9^\circ = \frac{a}{1}$



$x^2 + a^2 = 1^2$
$x^2 = 1^2 - a^2$
$x = \sqrt{1 - a^2}$
sin 18º
= sin 2(9º)
= 2 sin 9º.cos 9º
$= 2(a)(\sqrt{1 - a^2})$
$= 2a(\sqrt{1 - a^2})$

</td><td>

Learners need to recognise the similarity with previous questions done at Grade 11 level.
Secondly, learners need to notice that the information given is for the angle 9º, whereas the question is in relation to the angle 18º. Learners should start thinking about what the connection is between the two angle sizes.
Once all values for x, y and r are known, remind learners that these values are for a 9º angle.
Ask: *What statement can be made using 9º angles but linking it back to an 18º angle?* (Use the sine double angle identity).

</td></tr>
</table>

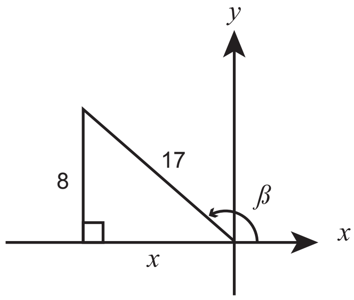
<table>
<tr><th>Example 2</th><th>Teaching notes</th></tr>
<tr><td>

Given sin $\beta = \frac{8}{17}$ where 90º ≤ β ≤ 270º, determine, without the use of a calculator:

a) cos 2β

b) sin 2β

Solution:



$x^2 + 8^2 = 17^2$

$x^2 = 289 - 64$

$x = \sqrt{225}$

$x = 15$

and $y = 8$; $r = 17$

a) $\cos 2\beta = 1 - 2\sin^2 A$

$= 1 - 2\left(\frac{8}{17}\right)^2$

$= 1 - 2\left(\frac{64}{289}\right)$

$= 1 - \frac{128}{289}$

$= \frac{161}{289}$

b) $\sin 2\beta = 2 \sin B . \cos B$

$= 2\left(\frac{8}{17}\right)\left(\frac{15}{17}\right)$

$= \frac{240}{289}$

</td><td>

Learners need to notice that the information given is for the angle β whereas the question is in relation to the angle 2β.

Remind learners that for this question they need to establish which quadrant to work in before using the theorem of Pythagoras to find the values of x, y and r.

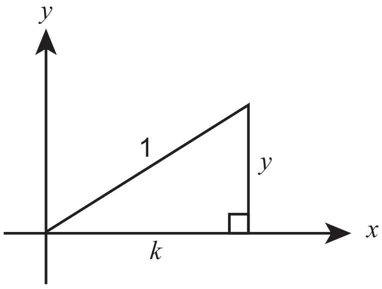
Once these are found, ask: *What statement can be made to link the 2β in the questions to the β in the information?*

(The double angle identities can be used).

Ask: *Which of the three versions of the cosine double identity should we use?*

(It does not matter – they will all give the same solution because they are all equal to each other).

</td></tr>
</table>

Example 3	Teaching notes
If cos $\theta = k$, and $\theta \in [0º;90º]$, determine the value of $\sin\left(\frac{\theta}{2}+30^0\right).\cos\left(\frac{\theta}{2}+30^0\right)$ with the aid of a diagram and without the use of a calculator. Solution:  $\cos\theta = \frac{k}{1}$ $y^2 + k^2 = 1^2$ $y^2 = 1 - k^2$ $y = \sqrt{1-k^2}$ $\sin\left(\frac{\theta}{2}+30^0\right).\cos\left(\frac{\theta}{2}+30^0\right)$ $= \frac{1}{2}\left[2\sin\left(\frac{\theta}{2}+30^0\right).\cos\left(\frac{\theta}{2}+30^0\right)\right]$ $= \frac{1}{2}[\sin(\theta+60^0)]$ $= \frac{1}{2}(\sin\theta.\cos 60^0 + \cos\theta.\sin 60^0)$ $= \frac{1}{2}\left(\left(\sqrt{1-k^2}\right)\left(\frac{1}{2}\right)+(k)\left(\frac{\sqrt{3}}{2}\right)\right)$ $= \frac{\sqrt{1-k^2}}{4}+\frac{\sqrt{3}k}{4}$ $= \frac{\sqrt{1-k^2}+\sqrt{3}k}{4}$	This question starts in the same way as the first example – a diagram is required and the values of x, y and r need to be found. Once this has been completed, point out again that the information found is linked to the angle θ and not $\frac{\theta}{2}$. Learners need to be able to make a statement linked to the one given that will have the angle θ in it. Ask: *What could we do?* (The statement almost looks like a double angle but is missing the '2' in front – we will need to 'force' a double angle).

10. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

11. Give learners an exercise to complete on their own.

12. Walk around the classroom as learners do the exercise. Support learners where necessary.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=kbkGgbvGo7M

TERM 1, TOPIC 4, LESSON 5

GENERAL SOLUTIONS

Suggested lesson duration: 1 hour

POLICY AND OUTCOMES

A

CAPS Page Number	42
Lesson Objectives By the end of the lesson, learners should be able to: ● find general solutions involving compound angles and double angles and factorising.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson have the three examples from point 2 ready.
5. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		CLASSROOM MATHS		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
7 8	129 131	5 6 7	97 99 100	5	127	6.4 6.5	129 130	5.9	147	4.4	160

C CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Learners have found the general solution in Grade 11 and revised it again in the revision lesson at the beginning of the section.
2. Although double and compound angles are often found within an equation, it is not usually a requirement to expand them. They just represent transformations in the function.
3. General solutions can be quite difficult for some learners. The examples chosen for this lesson are more advanced than those used before. Work through the examples slowly with learners.

DIRECT INSTRUCTION

1. Ask learners to find the general solution of the three examples below. Most learners should not require assistance at this stage.
2. Find the general solution:

$\sin 2x = 0{,}352$	$2 \tan \alpha = -1{,}35$	$\cos (\theta + 10°) = -0{,}584$

3. Once learners have had time to complete the examples, go through each example on the board. Learners should correct their work if any errors were made.

$\sin 2x = 0{,}352$	
$\sin 2x = 0{,}352$ Ref angle $(RA) = 20{,}61°$ Positive ratio ∴ Quadrant 1 and 2	
Quad 1: $2\theta = RA + k.360°$ $2\theta = 20{,}61° + k.360°$ $\theta = 10{,}31° + k.180°$	Quad 2: $2\theta = (180° - RA) + k.360°$ $2\theta = (180° - 20{,}61°) + k.360°$ $2\theta = 159{,}39° + k.360°$ $\theta = 79{,}7° + \text{k}.180°$
$k \in Z$	

$2 \tan \alpha = -1{,}35$

$2 \tan \alpha = -1{,}35$

$\tan \alpha = -0{,}675$

Ref angle $(RA) = 34{,}02^\circ$

Negative ratio $\therefore$ Quadrant 2 and 4

Quad 2:	Quad 4:
$\alpha = (180^\circ - RA) + k.180^\circ$	$\alpha = (360^\circ - RA) + k.180^\circ$
$\alpha = (180^\circ - 34{,}02^\circ) + k.180^\circ$	$\alpha = (360^\circ - 34{,}02^\circ) + k.180^\circ$
$\theta = 145{,}98^\circ + k.180^\circ$	$\theta = 325{,}98^\circ + k.180^\circ$

$k \in Z$

$\cos(\theta + 10^\circ) = -0{,}584$

$\cos(\theta + 10^\circ) = -0{,}584$

Ref angle $(RA) = 54{,}3^\circ$

Negative ratio $\therefore$ Quadrant 2 and 3

Quad 2:	Quad 3:
$\theta + 10^\circ = (180^\circ - \text{RA}) + k.360^\circ$	$\theta + 10^\circ = (180^\circ + \text{RA}) + k.360^\circ$
$\theta + 10^\circ = (180^\circ - 54{,}3^\circ) + k.360^\circ$	$\theta + 10^\circ = (180^\circ + 54{,}3^\circ) + k.360^\circ$
$\theta + 10^\circ = 125{,}7^\circ + k.360^\circ$	$\theta + 10^\circ = 234{,}3^\circ + k.360^\circ$
$\theta = 115{,}7^\circ + k.360^\circ$	$\theta = 224{,}3^\circ + k.360^\circ$

$k \in Z$

4. Ask learners to notice the double angle in the first equation and the compound angle in the last equation.
   Ask: *Did you use your knowledge of double and compound angles in solving the equation?* (No – the $2x$ in the first equation just meant a period change in the function and only affected the equation when needing to divide through by 2 to solve for x; the $(\beta + 10^\circ)$ in the third example is showing a horizontal shift in the function and only affected the equation when 30º needed subtracting from both sides to solve for β).

Note: It is important to link these equations back to functions. Learners seem to solve mathematical problems (often correctly) without realising what is actually being found.
The first equation is solving for the points of intersection of the graph $y = \sin 2x$ and the function $y = 0{,}352$ (a horizontal line.
The second equation is solving for the points of intersection of the graph $y = 2 \tan \alpha$ and the function $y = -1{,}35$
The third equation is solving for the points of intersection of the graph $y = \cos(\theta + 10^\circ)$ and the function $y = -0{,}584$
Draw these to demonstrate if necessary.

5. Do a few more fully worked examples of a more complicated nature. Some examples require the use of double angles and all of them require factorising.
   Learners should write the examples in their books, making notes as they do so.

Examples: Find the general solutions:	Teaching notes:

$$2\sin\beta.\cos\beta = \sin\beta$$

Many learners want to divide by sin β on both sides to solve this equation.
Ask: *Why can't we divide by sin β?*
(sin β could be equal to zero).
Learners need to get all the terms on one side equal to zero, then factorise and solve as with any other quadratic equation

$$\begin{aligned} 2\sin\beta.\cos\beta &= \sin\beta \\ 2\sin\beta.\cos\beta - \sin\beta &= 0 \\ \sin\beta(2\cos\beta - 1) &= 0 \end{aligned}$$

$$\begin{aligned} \sin\beta &= 0 \\ RA &= 0^\circ \end{aligned}$$

Quad 1:

$$\beta = k.360^\circ$$

Quad 2:

$$\beta = 180^\circ + k.360^\circ$$

$$\begin{aligned} 2\cos\beta - 1 &= 0 \\ \cos\beta &= \frac{1}{2} \\ RA &= 60^\circ \end{aligned}$$

Quad 1:

$$\beta = 60^\circ + k.360^\circ$$

Quad 4:

$$\begin{aligned} \beta &= 360^\circ - 60^\circ + \text{k}.360^\circ \\ \beta &= 300^\circ + k.360^\circ \end{aligned}$$

$$k \in Z$$

$$\cos 2\theta - \cos\theta + 1 = 0$$

Tell learners to note the double angle. The cosine of a double angle has three possible versions to use. When substitution is required, the version used is not important – the answer will be the same. In these types of questions, factorising will be easier if one particular version is used. Learners need to find the version that will lead to the simplest factorising. Tell learners to use all three, collect like terms if possible and tell you which one will factorise the easiest. Once this has been done, give learners the tip that if a '1' (+ or –), it is often the version that will eliminate that 1 that will work the best.

$$\cos 2\,\theta - \cos\theta + 1 = 0$$
$$2\cos^2\theta - 1 - \cos\theta + 1 = 0$$
$$2\cos^2\theta - \cos\theta = 0$$
$$\cos\theta\,(2\cos\theta - 1) = 0$$

$\cos\theta = 0$ or $2\cos\theta - 1 = 0$

$RA = 90^\circ$

Quad 1:

$\theta = 90^\circ + k.360^\circ$

Quad 4:

$\theta = 360^\circ - 90^\circ + k.360^\circ$

$\theta = 270^\circ + k.360^\circ$

$\cos\theta = \frac{1}{2}$

RA = 60º

Quad 1:

$\theta = 60^\circ + k.360^\circ$

Quad 4:

$\theta = 360^\circ - 60^\circ + k.360^\circ$

$\theta = 300^\circ + k.360^\circ$

$k \in Z$

$$\cos 2\theta - \cos\theta + 1 = 0$$

Tell learners to note the double angle. The cosine of a double angle has three possible versions to use. When substitution is required, the version used is not important – the answer will be the same. In these types of questions, factorising will be easier if one particular version is used. Learners need to find the version that will lead to the simplest factorising. Tell learners to use all three, collect like terms if possible and tell you which one will factorise the easiest. Once this has been done, give learners the tip that if a 1 (+ or –), it is often the version that will eliminate that 1 that will work the best.

$$2\cos^2 x - \cos x - \sin 2x + \sin x = 0$$
$$2\cos^2 x - \cos x - 2\sin x.\cos x + \sin x = 0$$
$$\cos x(2\cos x - 1) - \sin x(2\cos x - 1) = 0$$
$$(2\cos x - 1)(\cos x - \sin x) = 0$$

$2\cos x - 1 = 0$ or $\cos x - \sin x = 0$

$\cos x = \frac{1}{2}$

$RA = 60^\circ$

Quad 1:

$x = 60^\circ + k.360^\circ$

Quad 4:

$x = 360^\circ - 60^\circ + k.360^\circ$

$x = 300^\circ + k.360^\circ$

$\cos x = \sin x$

$\frac{\cos x}{\cos x} = \frac{\sin x}{\cos x}$

$1 = \tan x$

$RA = 45^\circ$

Quad 1:

$x = 45^\circ + k.180^\circ$

Quad 3:

$x = 180^\circ + 45^\circ + k.180^\circ$

$x = 225^\circ + k.180^\circ$

$k \in Z$

6. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

7. Give learners an exercise to complete on their own.
8. Walk around the classroom as learners do the exercise. Support learners where necessary.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=bI1AXR5kpr4

TERM 1, TOPIC 4, LESSON 6

IDENTITIES

Suggested lesson duration: 1 hour

POLICY AND OUTCOMES

A

CAPS Page Number	42
Lesson Objectives By the end of the lesson, learners should be able to: ● prove identities involving compound angles and double angles ● find the values of an unknown that make an identity invalid.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson have the two identities from point 2 ready.
5. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		CLASSROOM MATHS		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
6 9	126 132	4	94	4	123	5.4 5.7	115 122	5.7	142		

C CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Learners have already proved identities in grade 11 and found the values that would make an identity invalid.
2. In this lesson, the identities will be more challenging and also involve double angles.

DIRECT INSTRUCTION

1. Ask learners to prove the following identities. Although learners have done a number of these before, they often find these challenging, so may need assistance.
2. Prove the following identities:

$\frac{\cos\alpha}{1+\sin\alpha} + \tan\alpha = \frac{1}{\cos\alpha}$	$\frac{1 - \sin^3 x}{1 - \sin x} = 1 + \sin x + \sin^2 x$

3. Once learners have completed the examples, go through each one on the board. Learners should correct their work if any errors were made.

$\frac{\cos\alpha}{1+\sin\alpha} + \tan\alpha = \frac{1}{\cos\alpha}$ LHS $= \frac{\cos\alpha}{1+\sin\alpha} + \tan\alpha$ $= \frac{\cos\alpha}{1+\sin\alpha} + \frac{\sin\alpha}{\cos\alpha}$ $= \frac{\cos^2\alpha + \sin\alpha(1+\sin\alpha)}{(1+\sin\alpha)(\cos\alpha)}$ $= \frac{\cos^2\alpha + \sin\alpha + \sin^2\alpha}{(1+\sin\alpha)(\cos\alpha)}$ $= \frac{1+\sin\alpha}{(1+\sin\alpha)(\cos\alpha)}$ $= \frac{1}{\cos\alpha}$ = RHS	$\frac{1 - \sin^3 x}{1 - \sin x} = 1 + \sin x + \sin^2 x$ LHS $= \frac{1 - \sin^3 x}{1 - \sin x}$ $= \frac{(1 - \sin x)(1 + \sin x + \sin^2 x)}{1 - \sin x}$ $= 1 + \sin x + \sin^2 x$ = RHS

4. Ask learners what it means to have an invalid identity.
   Ask: *What part of an identity could cause it not to be valid?*
   (If tangent is involved it will be invalid for all the asymptotes; if denominators are involved, those trigonometric functions cannot equal zero).

5. Tell learners that you are going to show for what values of the variable the above identities would be invalid.

tan α: $\boldsymbol{\alpha = 90^\circ + k.180^\circ}$ 1 + sin α: $1 + \sin\alpha = 0$ $\sin\alpha = -1$ $RA = 90^\circ$ Quad 3: $\alpha = 180^\circ + 90^\circ + k.360^\circ$ $\alpha = 270^\circ + k.360^\circ$ Quad 4: $\alpha = 360^\circ - 90^\circ + k.360^\circ$ $\boldsymbol{\alpha = 270^\circ + k.360^\circ}$ $k \in Z$	cos α: $\cos\alpha = 0$ $RA = 90^\circ$ Quad 1: $\alpha = 90^\circ + k.360^\circ$ Quad 4: $\alpha = 360^\circ - 90^\circ + k.360^\circ$ $\alpha = 270^\circ + k.360^\circ$ $k \in Z$
1 – sin x: $1 - \sin x = 0$ $1 = \sin x$ RA = 90º	Quad 1: $x = 90^\circ + k.360^\circ$ Quad 2: $x = 180^\circ - 90^\circ + k.360^\circ$ $\boldsymbol{x = 90^\circ + k.360^\circ}$ $k \in Z$

6. Ask if there are any questions before moving on to proving identities that involve new concepts from Grade 12.

7. Do two fully worked examples with learners.
Learners should write the examples in their books and take notes as they do so.

$\dfrac{\cos\theta}{\sin\theta} - \dfrac{\cos 2\theta}{\sin\theta.\cos\theta} = \tan\theta$
Remind learners of the basics to follow when attempting to prove an identity. ● If there are fractions, find the lowest common denominator and subtract (or add if appropriate) accordingly. ● Consider the 'new' numerator – in this case there is a cosine double angle. ● Of the three versions that can be used, consider other terms within the numerator that will help to simplify it to become only one term like the right-hand side. Ask: *Which version could assist in simplifying the numerator to one term?* ($\cos^2\theta - \sin^2\theta$).

$$\frac{\cos\theta}{\sin\theta}-\frac{\cos 2\theta}{\sin\theta.\cos\theta} = \tan\theta$$

$$\begin{aligned}
\text{LHS} &= \frac{\cos\theta}{\sin\theta}-\frac{\cos 2\theta}{\sin\theta.\cos\theta}\\
&= \frac{\cos^2\theta-\cos 2\theta}{\sin\theta.\cos\theta}\\
&= \frac{\cos^2\theta-(\cos^2\theta-\sin^2\theta)}{\sin\theta.\cos\theta}\\
&= \frac{\cos^2\theta-\cos^2\theta+\sin^2\theta}{\sin\theta.\cos\theta}\\
&= \frac{\sin^2\theta}{\sin\theta.\cos\theta}\\
&= \frac{\sin\theta}{\cos\theta}\\
&= \tan\theta\\
&= \text{RHS}
\end{aligned}$$

$$\frac{\sin 2B}{1+\cos 2B} = \tan B$$

This identity also has the cosine double angle.

Ask: *Which version could assist in simplifying the numerator to one term?*

($2\cos^2 B - 1$)

Point out to learners that when ‘1’ is involved it is often a clue as to what version of the cosine double angle to use. Choose the version that will ‘remove’ the ‘1’.

$$\frac{\sin 2B}{1+\cos 2B} = \tan B$$

$$\begin{aligned}
\text{LHS} &= \frac{2\sin B.\cos B}{1+2\cos^2 B-1}\\
&= \frac{2\sin B.\cos B}{2\cos^2 B}\\
&= \frac{\sin B}{\cos B}\\
&= \tan B\\
&= \text{RHS}
\end{aligned}$$

8. Ask directed questions so that you can ascertain learners’ level of understanding.
   Ask learners if they have any questions.

9. Give learners an exercise to complete on their own.

10. Walk around the classroom as learners do the exercise. Support learners where necessary.

ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=Wh7hM_FsN3Y

https://www.youtube.com/watch?v=pQUq4YHW5v8

TERM 1, TOPIC 4, LESSON 7

REVISION AND CONSOLIDATION

Suggested lesson duration: 1 hour

A POLICY AND OUTCOMES

CAPS Page Number	42
Lesson Objectives By the end of the lesson, learners will have revised: ● all concepts of Trigonometry.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Have Resource 11 ready for use during this lesson.
4. Write the lesson heading on the board before learners arrive.
5. Write work on the chalkboard before the learners arrive. For this lesson have the first few questions ready.
6. The table below provides references to this topic in Grade 12 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**CLASSROOM MATHS**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
Rev S Ch	144 146	Rev	101	Qu’s	132	Rev	132	5.10 5.11	148 149	4.7 (1-16)	172

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Ask learners to recap what they have learned in this section. Point out issues that you know are important, as well as problems that you encountered from your own learners.
2. If learners want you to explain a concept again, do that now.

DIRECT INSTRUCTION

This lesson is made up of fully worked examples from a past examination covering most of the concepts in this topic. As you work through these with the learners, it is important to frequently talk about as many concepts as possible.
For example, use the words general solution, compound angles, double angles, functions, reductions and identities.

Say: *I am going to do an entire Trigonometry question from the 2016 final examination with you. You should write down the answers as I do them, taking notes at the same time.*

Questions

1. Given sin 16º = p, determine the following in terms of p, **without the use of a calculator**.
   a) sin 196º   b) cos 16º
2. Given cos (A – B) = cos A.cos B + sin A.sin B.
   Use the formula to derive a formula for sin (A + B)
3. Simplify $\dfrac{\sqrt{1 - \cos^2 2A}}{\cos(-A).\cos(90^0 + A)}$ completely, given that 0º < A < 90º
4. Given cos 2B = $\frac{3}{5}$ and 0º ≤ B ≤ 90º
   Determine, **without using a calculator**, the value of EACH of the following in its simplest form:
   a) cos B
   b) sin B
   c) cos (B + 45º)

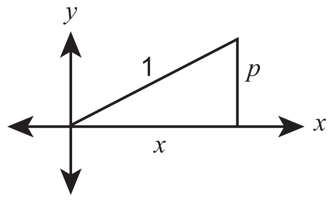
TOPIC 4, LESSON 7: REVISION AND CONSOLIDATION

Teaching notes:

1. Ask learners to list the steps to follow with this type of question. Prompt where necessary.
   - Make the variable into a fraction (put it over 1).
   - Draw a right-angled triangle. Fill in the angle and the two know sides from the fraction
   - Use Pythagoras to find the third side in terms of the variable
   - Reduce where necessary and substitute
2. This is bookwork. Remind learners that co-ratios will be required.
3. This question requires knowledge of the basic identity $\sin^2\theta + \cos^2\theta = 1$, as well as the various versions of it. Knowledge of co-ratios and negative angles is also required. Question learners regarding all these concepts.
4. Learners need to recognise that the given information is in terms of the angle $2B$ whereas the questions are in terms of B. This should make them aware that double angles will be used.
   Learners may need to be reminded how to rationalise the denominator.
   For (c), the compound formulae will be required.

Solutions

1. $\sin 16º = \frac{p}{1}$

	$x^2 + p^2 = 1^2$ $x = \sqrt{1 - p^2}$ and $r = 1$ $y = p$	a) sin 196º = sin (180º + 16º) = –sin 16º = $-p$ b) cos 16º = $\sqrt{1 - p^2}$

2. sin $(A + B)$ = cos $[90º – (A + B)]$
   sin $(A + B)$ = cos $[90º – A – B]$
   sin $(A + B)$ = cos $[(90º – A) – B]$
   sin $(A + B)$ = cos $(90º – A)$.cos B + sin $(90º – A)$.sin B
   sin $(A + B)$ = sin A.sin B + cos A.cos B

3.
$$\frac{\sqrt{1-\cos^2 2A}}{\cos(-A).\cos(90^0+A)} = \frac{\sqrt{\sin^2 2A}}{\cos A - \sin A}$$
$$= \frac{\sin 2A}{-\cos A.\sin A}$$
$$= \frac{2\sin A.\cos A}{-\cos A.\sin A}$$
$$= -2$$

4. $\cos 2B = \frac{3}{5}$

a) $2\cos^2 B - 1 = \frac{3}{5}$

$$2\cos^2 B = \frac{3}{5} + 1$$

$$2\cos^2 B = \frac{8}{5}$$

$$\cos^2 B = \frac{4}{5}$$

$$\cos B = \pm\sqrt{\frac{4}{5}} = \pm\frac{2}{\sqrt{5}}$$

$$= \frac{2\sqrt{5}}{5}$$

b) $1 - 2\sin^2 B = \frac{3}{5}$

$$-2\sin^2 B = \frac{3}{5} - 1$$

$$-2\sin^2 B = -\frac{2}{5}$$

$$\sin^2 B = \frac{1}{5}$$

$$\sin B = \pm\sqrt{\frac{1}{5}} = \pm\frac{1}{\sqrt{5}}$$

$$= \frac{\sqrt{5}}{5}$$

c) $\cos(B + 45^\circ) = \cos B.\cos 45^\circ - \sin B.\sin 45^\circ$

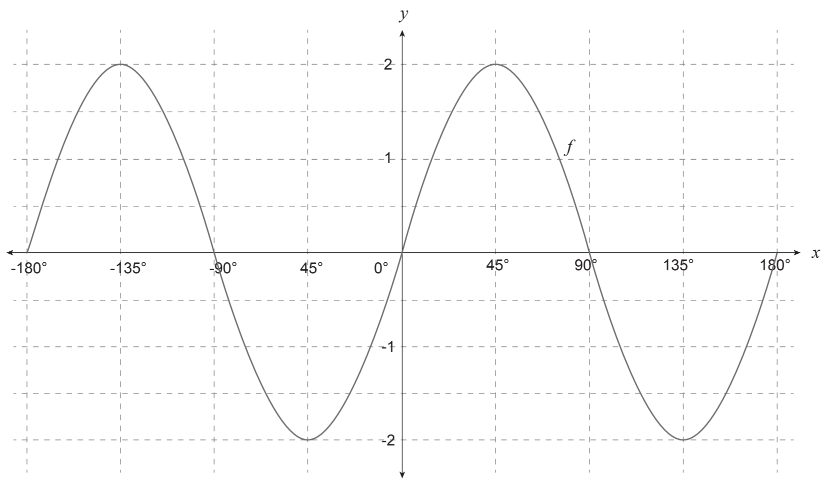
$$\cos(B + 45^\circ) = \frac{2\sqrt{5}}{5}.\frac{\sqrt{2}}{2} - \frac{\sqrt{5}}{5}.\frac{\sqrt{2}}{2}$$

$$\cos(B + 45^\circ) = \frac{2\sqrt{10}}{10} - \frac{\sqrt{10}}{10}$$

$$= \frac{\sqrt{10}}{10}$$

Questions

In the diagram the graph of $f(x) = 2\sin 2x$ is drawn for the interval $x \in [-180^\circ ; 180^\circ]$



TOPIC 4, LESSON 7: REVISION AND CONSOLIDATION

1. On the system of axes on which f is drawn, draw the graph of $g(x) = –\cos 2x$ for x [–180º ; 180º], clearly showing all intercepts with the axes, the coordinates of the turning points and end points of the graph.
2. Write down the maximum value of $f(x) – 3$
3. Determine the general solution of $f(x) = g(x)$
4. Hence, determine the values of x for which $f(x) = g(x)$ in the interval x [–180º ; 0º]

Teaching notes:

1. Discuss what the '–' and the 2 in front of the x do to the basic cosine function.
   The '–' will reflect the graph in the x-axis. The '2' changes the period of the graph.
   The new period is $\frac{360^0}{2}$ = 180º.
2. Ask: *What transformation does* '–3' *perform on the function?*
   (It moves down 3 units).
   Ask: *What was the maximum value of the original function?*
   (2).
3. Ask: *How should we approach an equation that has both sine and cosine in it and the angles are the same?*
   (Divide both sides by the cosine of the angle to change the equation into a tangent.
   Ask: *What is required to start solving the equation?*
   (Reference angle).

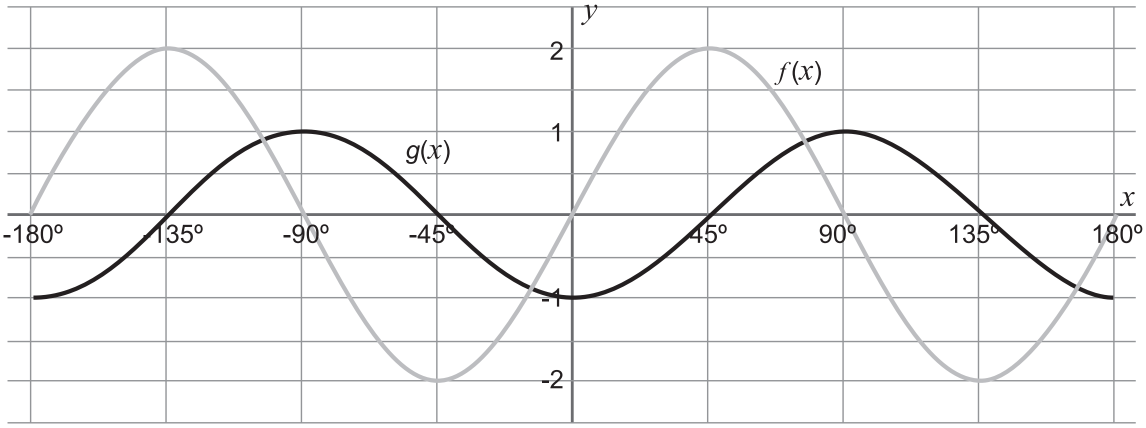
Remind learners of the steps to follow if necessary:

- Use the 2nd function on the calculator: (shift ; trig function ; ratio) to find the reference angle
- Note whether the function is positive or negative
- Choose the quadrants accordingly and find the general solutions according to the quadrants
- Use the appropriate reductions to represent angles in the chosen quadrants.

4. Ask: *How can the general solution be used to find actual solutions?*
   (Use $k \in Z$ and substitute appropriate values).
   Tell learners to note the values of the x given and to use the sketch to check their answers.

Solutions

1.



2. The maximum value of $f(x) = 2$; ∴ the maximum value of $f(x) - 3 = -1$

3. $f(x) = g(x)$

$$2 \sin 2x = -\cos 2x$$

$$\frac{2 \sin 2x}{\cos 2x} = \frac{-\cos 2x}{\cos 2x}$$

$$2 \tan 2x = -1$$

$$\tan 2x = -\frac{1}{2}$$

$$RA = 26{,}57^\circ$$

Quad 2	Quad 4
$2x = 180^\circ - 26{,}57^\circ + k.180^\circ$	$2x = 360^\circ - 26{,}57^\circ + k.180^\circ$
$2x = 153{,}43^\circ + k.180^\circ$	$2x = 333{,}43^\circ + k.180^\circ$
$x = 76{,}72^\circ + k.90^\circ$	$x = 166{,}72^\circ + k.90^\circ$ $k \in Z$

4. $x \in \{-103{,}28^\circ ; -13{,}28^\circ\}$

1. Ask directed so that you can ascertain learners' level of understanding.
   Ask learners if they have any questions.

2. Give learners an exercise to complete on their own.

3. Walk around the classroom as learners do the exercise. Support learners where necessary.